RAK-33060 Fracture mechanics and fatigue

4. Exercise, energy principles weight functions

Problem 1. Use the principle of virtual work to solve the following beam problem

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = q, \quad x \in (0, L),$$

with boundary conditions $v(0) = v'(0) = 0$, $v(L) = \Delta$ and $M(L) = 0$. For the prescribed displacement Δ choose $\Delta = \alpha q L^4/(12EI)$, where α is a dimensionless parameter.

1. Construct the simplest possible trial function for $v(x)$.
2. Construct the corresponding virtual displacement $\delta v(x)$.
3. Solve the problem and plot the deflection v and the bending moment $M(x)$ for $\alpha = 0.02$.

Solution. Requirements for the tria function is that the strain energy calculatex from it should be finite and the essential boundary conditions, i.e. the displacement boundary conditions should be satisfied. Thus, let us try the following polynomial form

$$v(x) = A(x/L)^2 + B(X/L)^3,$$

where A and B are unknown parameters. It can be easily noticed that the essential boundary conditions at $x = 0$ are satisfied i.e. $v(0) = v'(0) = 0$. At $x = L$ there is non-homogeneous essential boundary condition $v(L) = \Delta$, which results in

$$v(L) = A + B = \Delta, \quad \Rightarrow \quad B = \Delta - A.$$

Therefore, there is only one unknown parameter A and

$$v(x) = \left[(x/L)^2 - (x/L)^3\right] A + (X/L)^3\Delta.$$

For the virtual displacement δv we can take

$$\delta v(x) = (x/L)^2 C + (x/L)^3 D.$$

Since the virtual displacement has to satisfy *homogeneous* essential boundary conditions where the true displacement has essential boundary conditions, thus

$$\delta v(L) = 0, \quad \Rightarrow \quad D = -C.$$

The principle of virtual work for the Euler-Bernoulli beam model states that for all kinematically admissible virtual displacements the total virtual work has to vanish

$$-\int_0^L \delta v'' EI v \,\mathrm{d}x + \int_0^L \delta v \,\mathrm{d}x = 0. \tag{1}$$

The second derivatives of the displacement and virtual displacement are

$$v'' = \left(2/L^2 - 6x/L^3\right) A + 6(x/L^3)\Delta, \quad \delta v'' = (2/L^2)(1 - 3x/L)C.$$

Inserting the trial function and the virtual displacement to the virtual work equation (1) gives

$$C(-KA + F) = 0.$$

Since C is arbitrary, we have

$$KA = F,$$

where

$$K = \int_0^L \frac{4EI}{L^4}\left(1 - \frac{3x}{L}\right)^2 \mathrm{d}x, \quad F = \int_0^L q\left(\frac{x}{L}\right)^2\left(1 - \frac{3x}{L}\right)\mathrm{d}x - \int_0^L \frac{12EI}{L^4}\left(\frac{x}{L}\right)\left(1 - \frac{3x}{L}\right)\mathrm{d}x\Delta.$$

Integrating (numerical quadrature can be used, e.g. Simpson's rule), gives

$$K = \frac{4EI}{L^3}, \quad F = \frac{EI}{L^2}\left(\frac{1}{12}\xi + 6\alpha\right),$$

where we have defined dimensionless quantities ξ and α as

$$\alpha = \frac{\Delta}{L}, \quad \xi = \frac{qL^3}{EI}.$$

Solution for the unknown parameter A is

$$A = \frac{L}{4}\left(\frac{1}{12}\xi + 6\alpha\right).$$

Then

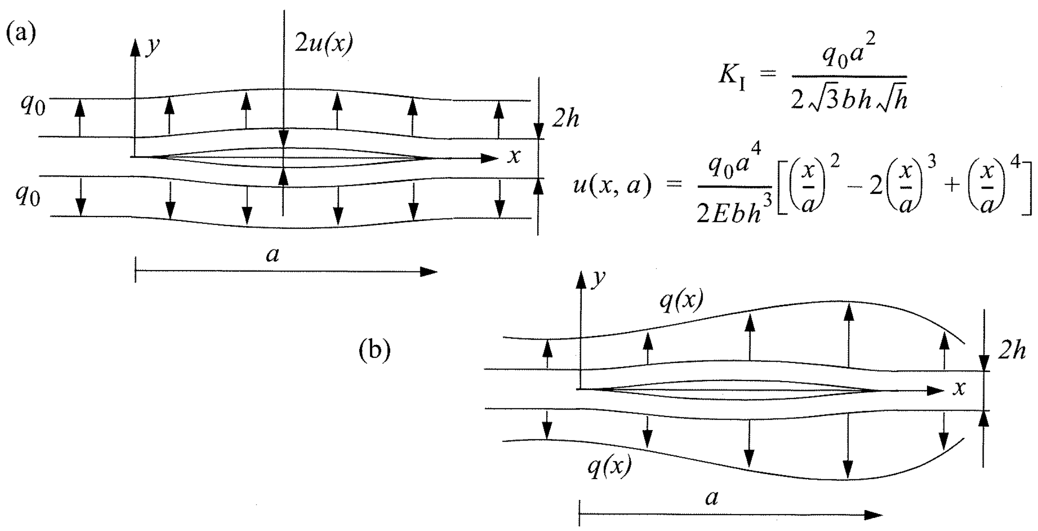
$$B = \Delta - A = -\frac{1}{2}\left(\alpha + \frac{1}{24}\xi\right)L,$$

and

$$v(x) = \left[\frac{\xi}{48}\left(\frac{x}{L}\right)^2\left(1 - \frac{x}{L}\right) + \frac{\alpha}{2}\left(\frac{x}{L}\right)^2\left(3 - \frac{x}{L}\right)\right]L.$$

Bending moment is $M = -EIv''$ ..., draw the figures.

Problem 2. Determine the weight function $h(x)$ and give the expression for the stress-intensity factor of the right crack tip in the split beam loaded with arbitrarily distributed load $q(x)$ (force/length) according to the figure (b). Width of the beam is b. Utilise the solution to the problem in figure a(and assume plane stress conditions. Calculate K_{I} for the case $q(x) = q_0 x/a$.



Solution. Equation for the weight function h is:

$$h(x_i) = \frac{E}{2K_I^{(1)}} \frac{\partial u_i^{(1)}}{\partial a}.$$

And according to the given equations of u_i and K :

$$h(x) = \frac{E}{2K_I^{(1)}} \frac{q_0}{2Ebh^3} \frac{\partial}{\partial a}(a^2x^2 - 2ax^3 + x^4).$$

Which gives:

$$h(x) = \frac{\sqrt{3}a}{bh\sqrt{h}}(\frac{x}{a})^2(1 - \frac{x}{a}).$$

The stress intensity factor is given by

$$K_{\mathrm{I}}^{(2)} = \int_{\Gamma_c} q(x)h(x)dx.$$

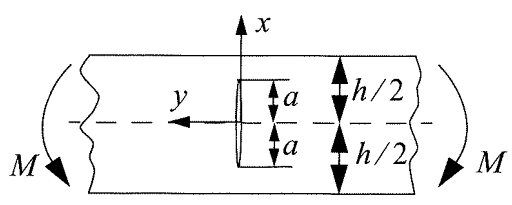
So

$$K_{\mathrm{I}}^{(2)} = \frac{\sqrt{3}a}{bh\sqrt{h}} 2\int_0^a q(x)(\frac{x}{a})^2(1 - \frac{x}{a})dx.$$

For the loading, $q(x) = q_0\dfrac{x}{a}$

$$\begin{aligned} K_{\mathrm{I}}^{(2)} &= \frac{2\sqrt{3}a}{bh\sqrt{h}} \int_0^a q_0(\frac{x}{a})^3(1 - \frac{x}{a})dx \\ &= \frac{2\sqrt{3}aq_0}{bh\sqrt{h}}.(\frac{1}{4}\frac{x^4}{a^3} - \frac{1}{5}\frac{x^5}{a^4})\Big|_0^a \\ &= \frac{2\sqrt{3}a^2q_0}{bh\sqrt{h}}(\frac{1}{4} - \frac{1}{5}) = \frac{\sqrt{3}}{10}\frac{q_0a^2}{bh\sqrt{h}}. \end{aligned}$$

Problem 3. A beam of width b has a crack according to the figure below and is loaded in pure bending. Determine the stress-intensity factor when $h \gg 2a$. The material is linearly elastic.



Solution. The stress caused by the bending moment is given by:

$$\sigma = \frac{Mx}{I} \qquad \text{and} \qquad I = \frac{1}{12}bh^3.$$

So:

$$\sigma = \frac{12M}{bh^3}x.$$

The intensity factor equation is given by:

$$K_I = \int_{-a}^{a} \sigma(x) h(x) dx.$$

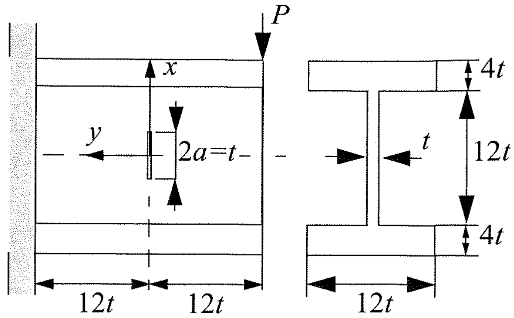
The weight function is:

$$h(x) = \frac{1}{\sqrt{\pi a}} \sqrt{\frac{a+x}{a-x}}.$$

Thus:

$$K_I = \frac{1}{\sqrt{\pi a}} \frac{12M}{bh^3} \frac{\pi a^2}{2} = \frac{6Ma^{\frac{3}{2}}}{bh^3} \sqrt{\pi}.$$

Problem 4. A short cantilever beam of a linearly elastic material contains a crack oriented according to the figure below. Compute an approximate expression for the stress intensity factor if the height of the web of the I-beam can be regarded as much larger than the crack size.



Solution. The bending stress is $\sigma_y(x) = \dfrac{12Pt}{I}x$ where $I = 6416t^4$,
and the shear stress: $\tau_{xy} = \dfrac{PS}{It}$.
The weight function is: $h^{(1)}(x) = \dfrac{1}{t\sqrt{\pi a}}\sqrt{\dfrac{a+x}{a-x}}$.
And the intensity factor will be:

$$K_I^{(2)} = t\int_{-a}^{a} \sigma_y(x)h^{(1)}(x)dx.$$

By replacing the values:

$$\begin{aligned}
K_I^{(2)} &= \frac{1}{\sqrt{\pi a}}\int_{-a}^{a} \sigma_y(x)\sqrt{\frac{a+x}{a-x}}dx \\
&= \sqrt{\left(\frac{2}{\pi t}\right)}\frac{12Pt}{6416t^4}\int_{-t/2}^{t/2}\sqrt{\frac{a+x}{a-x}}dx \\
&= \sqrt{\frac{\pi}{2}}\frac{3P}{6416t^{\frac{3}{2}}}.
\end{aligned}$$

For mode II also the we have:

$$K_{II} = t\int_{-a}^{a} \tau_{xy}h^{(1)}(x)dx.$$

And the static moment wrt the neutral axis S has the value

$$S = \sum A'y' = (4t)(12t)(8t) + (6t)(t)(3t) = 402t^3.$$

So:

$$K_{II} = t\int_{-a}^{a} \tau_{xy}\frac{1}{t\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}}dx = \frac{201P}{6416}\frac{1}{2t^{\frac{1}{2}}}\sqrt{\frac{\pi}{2}}.$$