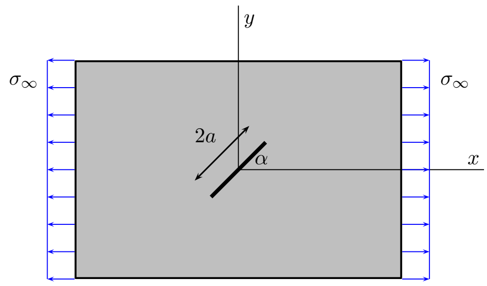
RAK-33060 Fracture mechanics and fatigue

3. Exercise

Problem 1. A large plate has a crack inclined by an angle α w.r.t. the horizontal line. The length of the crack is $2a$. The plate is loaded by a horizontal tensile stress $\sigma_x = \sigma_\infty$. Determine the stress intensity factors at the crack tip. At the end of this paper there is a table of stress intensity factors for basic loading cases.



Solution. The stress state perpendicular and parallel to the crack faces is

$$\sigma_\alpha = \sigma_\infty \sin^2\alpha, \qquad \tau_\alpha = \sigma_\infty \sin\alpha\cos\alpha.$$

Thus

$$K_{\rm I} = \sigma_\infty \sin^2\alpha\sqrt{\pi a}, \quad \text{and} \quad K_{\rm II} = \sigma_\infty \sin\alpha\cos\alpha\sqrt{\pi a}.$$

Problem 2. Investigate the previous structure. Assume that the fracture occurs if

$$\left(\frac{K_{\rm I}}{K_{\rm Ic}}\right)^2 + \left(\frac{K_{\rm II}}{K_{\rm IIc}}\right)^2 = 1, \tag{1}$$

where $K_{\rm Ic} \neq K_{\rm IIc}$. Investigate which angles α are the most dangerous as a function of the ratio $K_{\rm Ic}/K_{\rm IIc}$.

Solution. The stress intensity factors are based on the previous question

$$K_{\rm I} = \sigma_\infty \sin^2\alpha\sqrt{\pi a}, \quad \text{ja} \quad K_{\rm II} = \sigma_\infty \sin\alpha\cos\alpha\sqrt{\pi a}.$$

Replace them in condition(1), Thus, this is obtained:

$$(K_{\rm IIc}^2\sigma_\infty^2\sin^4\alpha + K_{\rm Ic}^2\sigma_\infty^2\sin^2\alpha\cos^2\alpha)\pi a = K_{\rm Ic}^2K_{\rm IIc}^2$$

The stress σ_∞ can be solved

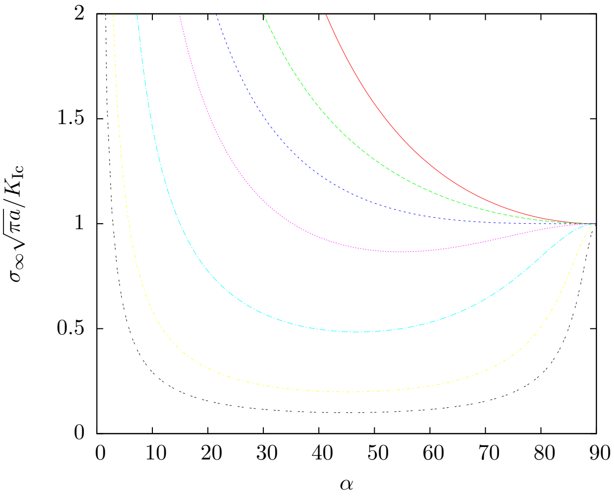
$$\sigma_\infty^2 = \frac{K_{\rm Ic}^2K_{\rm IIc}^2}{\pi a(K_{\rm IIc}^2\sin^4\alpha + K_{\rm Ic}^2\sin^2\cos^2\alpha)} = \frac{K_{\rm IIc}^2}{\pi a(\sin^2\alpha\cos^2\alpha + \beta^2\sin^4\alpha)} \tag{2}$$

$$= \frac{\beta^2K_{\rm Ic}^2}{\pi a(\sin^2\alpha\cos^2\alpha + \beta^2\sin^4\alpha)}, \tag{3}$$

where $\beta = K_{\rm IIc}/K_{\rm Ic}$. The dimensionless form is

$$\frac{\sigma_\infty\sqrt{\pi a}}{K_{\rm Ic}} = \frac{\beta}{\sqrt{\sin^2\alpha\cos^2\alpha + \beta^2\sin^4\alpha}}. \tag{4}$$

Stress graph below is shown with different values of β:n. Curves from top to bottom have respectively β:n values $2, 1, 1/\sqrt{2}, 0.5, 0.25, 0.1, 0.05$.



The stress σ_∞ reaches the minimum value when the denominator has the maximum value. Thus the investigated function is

$$f(\alpha) = \sin^2\alpha\cos^2\alpha - \beta^2\sin^4\alpha = (\beta^2 - 1)\sin^4\alpha + \sin^2\alpha.$$

The condition for the existance of the extreme value is to equal $f'(\alpha)$ to zero:

$$\begin{aligned} f'(\alpha) &= 4(\beta^2 - 1)\sin^3\alpha\cos\alpha + 2\sin\alpha\cos\alpha \\ &= 2\sin\alpha\cos\alpha[1 - 2(1 - \beta^2)\sin^2\alpha] = 0. \end{aligned}$$

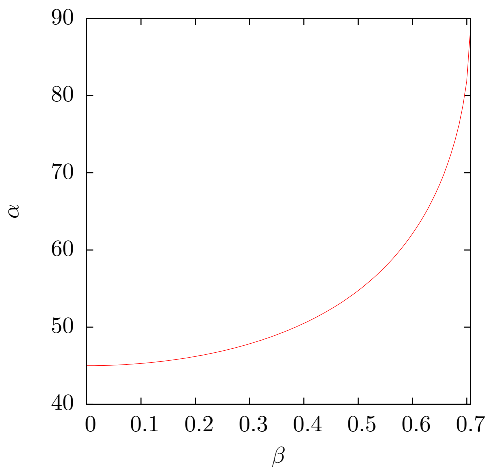
Thus

$$\sin\alpha = 0 \quad \text{tai} \quad \cos\alpha = 0 \quad \text{tai} \quad \sin^2\alpha = \frac{1}{2(1-\beta^2)}.$$

because $\sin^2\alpha \le 1$,so $\beta^2 \le \frac{1}{2}$. If $\sin\alpha = 0$ the solution is $\alpha = 0, \pi$,which are not relevant answers. If $\cos\alpha = 0$ solution is $\alpha = \pm\pi/2$, which corresponds to opening mode. For the third answer the angle value is

$$\alpha = \arcsin\frac{1}{\sqrt{2(1-\beta^2)}},$$

For which the graph is plotted below.



In this case equation (2) will be

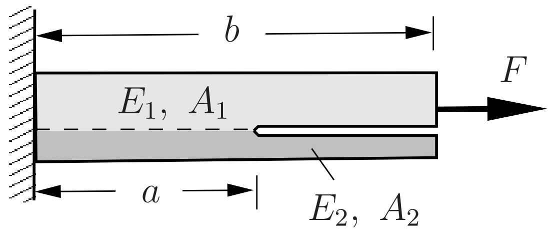
$$\frac{\sigma_\infty^2 \pi a}{K_{\mathrm{Ic}}^2} = \frac{K_{\mathrm{IIc}}^2}{K_{\mathrm{Ic}}^2(\sin^2\alpha\cos^2\alpha + \beta^2\sin^4\alpha)} = \frac{\beta^2}{\sin^2\alpha[1-(1-\beta^2)\sin^2\alpha]} = 4\beta^2(1-\beta^2). \tag{5}$$

That gives

$$\frac{\sigma_\infty\sqrt{\pi a}}{K_{\mathrm{Ic}}} = 2\beta\sqrt{1-\beta^2}. \tag{6}$$

Problem 3. A crack grows along the interface in a bi-material bar of width B under a tensile force F.

1. Determine the crack driving force $\mathcal{G}$ using simple bar model.
2. Determine K_{II} for the case $E_1 = E_2$ under the assumption that pue mode II and plane stress is present.



Solution. Consider the length of crack $c = b - a$. Find the compliance function equation by assuming that stress at the intact end of rod is equally distributed on cross section area $A_1 + A_2$ and cross section area A_1 with crack. Thus the copliance function is C on

$$C = \frac{c}{E_1A_1} + \frac{b-c}{E_1A_1 + E_2A_2}.$$

The energy release rate is

$$\mathcal{G} = -\frac{\mathrm{d}\Pi}{\mathrm{d}c} = \frac{F^2}{2B}\frac{\mathrm{d}C}{\mathrm{d}c}.$$

Now

$$\frac{\mathrm{d}C}{\mathrm{d}c} = \frac{1}{E_1A_1} - \frac{1}{E_1A_1+E_2A_2} = \frac{E_2A_2}{E_1A_1(E_1A_1+E_2A_2)},$$

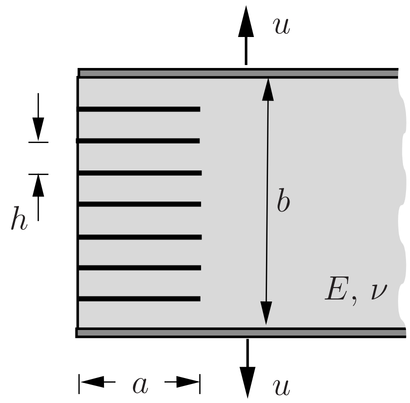
Thus

$$\mathcal{G} = \frac{F^2E_2A_2}{2BE_1A_1(E_1A_1+E_2A_2)}.$$

If we consider $E_1 = E_2$, the stress intensity factor is

$$K_{\mathrm{II}} = \sqrt{\mathcal{G}E} = \sqrt{\frac{A_2}{2BA_1(A_1+A_2)}}F.$$

Problem 4. Calculate the crack driving force $\mathcal{G}$ and the stress intensity factor K_{I} for the structure shown below. Assume the state of plane strain and that $h \ll a$.

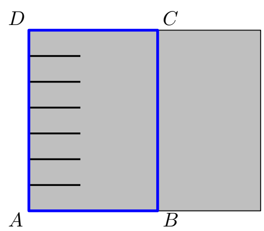


Solution. The deformation equations are

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_y}{E} = 0 \quad \Rightarrow \quad \sigma_z = \nu\sigma_y,$$
$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} = \frac{1-\nu^2}{E}\sigma_y = \frac{2u}{b}.$$

J-integral is

$$J = \int_C (Un_x + t_i u_{i,x})\mathrm{d}s = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}.$$



On the edge AB, normal vector is $(n_x, n_y, n_z)^T = (0, -1, 0)^T$, thus $n_x = 0$. The term $t_i u_{i,x} = t_x u_{x,x} + t_y u_{y,x} = \tau_{yx} u_{x,x} + \sigma_y u_{y,x}$, now $\tau_{yx} = 0$ and $u_{y,x} = 0$, so the edge AB has no effect.

On the edge BC, the normal vector is $(n_x, n_y, n_z)^T = (1, 0, 0)^T$, thus $n_x = 1$. The term $t_i u_{i,x} = t_x u_{x,x} + t_y u_{y,x} = \sigma_x u_{x,x} + \tau_{xy} u_{y,x}$, now $\sigma_x = 0$ and $\tau_{xy} = 0$. The strain energy on the edge BC is

$$U = \frac{1}{2}(\sigma_y\varepsilon_y + \sigma_z\varepsilon_z) = \frac{1}{2}\sigma_y\varepsilon_y = \frac{2E}{1-\nu^2}\frac{u^2}{b^2}$$

So

$$\int_{BC} = \int_0^b U\mathrm{d}y = \frac{2E}{1-\nu^2}\frac{u^2}{b}.$$

The edge CD is the same as AB. The edge DA ha no tension, so it has no effect.

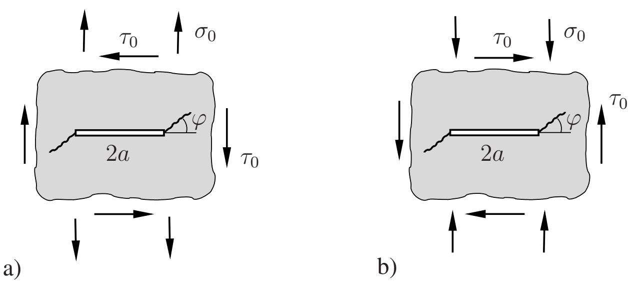
So the J-integral is

$$J = \int_{BC} = \int_0^b U\mathrm{d}y = \frac{2E}{1-\nu^2}\frac{u^2}{b}.$$

The structutre has n-pieces of cracks, $n = (b-h)/h$, when $\mathcal{G} = J$, it gives

$$K_\mathrm{I} = \sqrt{\mathrm{G}E} = Eu\sqrt{\frac{2h}{(1-\nu^2)b(b-h)}}.$$

Problem 5. Calculate the crack deflection angle φ for the two configurations shown below. Use the criterion of maximum circumferential stress and assume $\tau_0 = \sigma_0/2$.



Solution. The circumferential stress is obtained by the combination of mode I and II

$$\sigma_\phi = \frac{1}{4\sqrt{2\pi r}}\left[K_{\rm I}(3\cos(\phi/2) + \cos(3\phi/2)) - K_{\rm II}(3\sin(\phi/2) + 3\sin(3\phi/2))\right].$$

This can be in form

$$\sigma_\phi = \frac{1}{\sqrt{2\pi r}}\left[K_{\rm I}\cos(\phi/2)\frac{1+\cos\phi}{2} - \frac{3}{2}K_{\rm II}\sin\phi\cos(\phi/2)\right], \tag{7}$$

The exterme value is obtained when

$$K_{\rm I}\sin\phi + K_{\rm II}(3\cos\phi - 1) = 0. \tag{8}$$

Suppose that the solution of equation (8) results ϕ_0. The exteme value can be obtained by another differentiation at the stress on that point. The condition for the existence of maximum is that

$$\frac{\partial^2\sigma_\phi}{\partial\phi^2}_{|\phi=\phi_0} < 0.$$

The stress intensity factors according to the table

$$K_{\rm I} = \sigma_0\sqrt{\pi a}, \quad K_{\rm II} = \tau_0\sqrt{\pi a},$$

Now $\tau_0 = \sigma_0/2$. The maximum stress is reached at point ϕ_0 which fulfils

$$\sigma_0\sqrt{\pi a}\left[\sin\phi_0 + \tfrac{1}{2}(3\cos\phi_0 - 1)\right],$$

Figure (b) results in $\cos\phi_0 = 1/3$, and the angle value is $\phi_0 = \pm 70,6^\circ$. Which one is chosen depends on whether it is the minimum or the maximum, so the second derivative with respect to ϕ can give us the result. which now, the second derivation of the maximum value should be negetive. It follows that the negetive one is selected as the maximum answer.

Figure (a), the angle is obtained by

$$\sin\phi_0 - \tfrac{3}{2}\cos\phi_0 + \frac{1}{2} = 0,$$

It follows

$$\sin^2\phi_0 + \frac{4}{13}\sin\phi_0 - \frac{8}{13} = 0,$$

Which is obtained

$$\sin\phi_0 = -\frac{2}{13} \pm \frac{6\sqrt{3}}{13}, \quad \text{ja + juuri on oikea} \quad \phi_0 \approx 40.2^\circ.$$