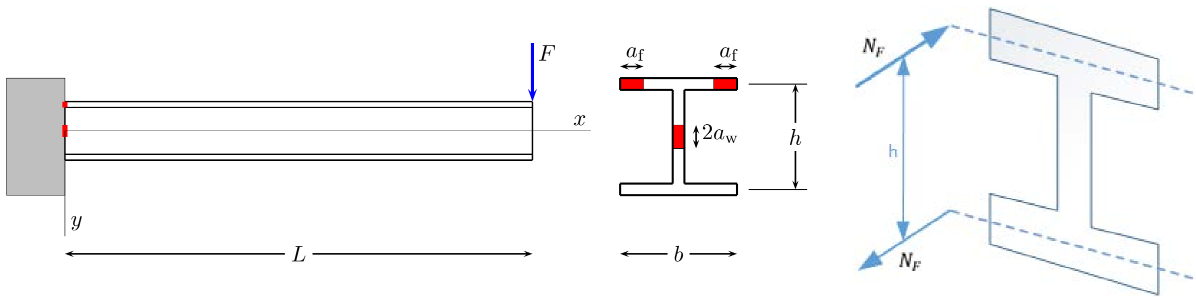
RAK-33060 Fracture mechanics and fatigue

2. Exercise

Problem 1. A tip loaded cantilever I-beam has cracks at the clamped end. The cross-section can be considered as an ideal I-profile. In the web there is a crack of length $2a_\mathrm{w} = 2t$ positioned symmetrically about the neutral axis. At the flange tips there are two symmetrically positioned cracks. How long (a_f =?) these flange tip cracks should be in order to be more dangerous than the crack in the web? Thickness of the web is t and the flanges $3t/2$, respectively. The other dimensions are related as $L/h = 10$, $h/t = 50$ and $b = h/2$. The fracture toughness in the mode II is $K_\mathrm{IIc} = (\sqrt{3}/2)K_\mathrm{Ic}$, where K_Ic is the mode I fracture toughness.

You can assume that the shear stresses are distributed uniformly in the web. As the cross-section is assumed to be an ideal I-section, the moment of inertia for the web can be neglected. The bending stresses can also be assumed to have a constant value in the flanges.



Solution. The absolute value of the bending moment at the clamped end is $M = FL$, and assuming an ideal I-profile, the resultant of the stresses at flanges gives $M = \sigma\frac{3}{2}tbh$ (*), from which we get

$$\sigma = \frac{2F}{3tb}\frac{L}{h}.$$

The stress intensity factor for the crack in the flange is $K_\mathrm{I} = f\sigma\sqrt{\pi a_\mathrm{f}}$, where $f = 1,12$.

The shear stress in the web is $\tau = F/th$ and the stress intensity factor $K_\mathrm{II} = \tau\sqrt{\pi a_\mathrm{w}}$, where now $a_\mathrm{a} = t$.

The condition asked in the problem is therefore

$$\frac{K_\mathrm{I}}{K_\mathrm{Ic}} > \frac{K_\mathrm{II}}{K_\mathrm{IIc}},$$

from which we get

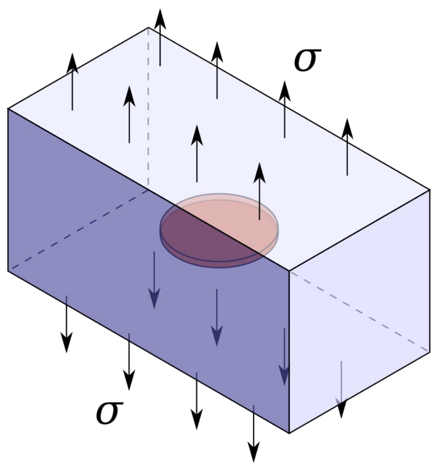
$$\frac{f\frac{2F}{3tb}\frac{L}{h}\sqrt{\pi a_\mathrm{f}}}{\frac{F}{th}\sqrt{\pi a_\mathrm{w}}} > \frac{K_\mathrm{Ic}}{\frac{\sqrt{3}}{2}K_\mathrm{Ic}},$$

and further

$$a_\mathrm{f} > 3\left(\frac{b}{L}\right)^2\frac{t}{f^2} \approx 6\cdot 10^{-3}t.$$

(*) If the normal stresses are assumed to be constant in the flanges and neglecting the effect of normal stresses in the web, we can compute the moment easily by consideting the resultant force in the flange $N_\mathrm{F} = \sigma b\frac{3}{2}t$ and then the moment is $M = N_\mathrm{F}h = \frac{3}{2}\sigma tbh$.

Problem 2. Determine the stress intensity factor K_{I} for a penny-shaped crack of radius a in an infinite domain under uniaxial stress σ. Use the Griffith energy approach assuming that stresses are relaxed in a ball of radius a around the crack. Compare to the values you have found in the literature.



Solution. Assume that stresses vanish in a ball around the crack. Thus the potential energy is reduced to a value

$$\Pi = \Pi_0 - \frac{4}{3}\pi a^3 \left(\frac{1}{2}\sigma\varepsilon\right) = \Pi_0 - \frac{4}{3}\pi a^3 \left(\frac{1}{2}\frac{\sigma^2}{E}\right). \tag{1}$$

Area of the crack is $A = \pi a^2$, thus

$$a = \sqrt{A/\pi}. \tag{2}$$

The crack driving force is

$$\mathcal{G} = -\frac{\mathrm{d}\Pi}{\mathrm{d}A} = \frac{\mathrm{d}}{\mathrm{d}A}\left(\frac{4}{3}\pi\left(\frac{A}{\pi}\right)^{3/2}\right)\left(\frac{1}{2}\frac{\sigma^2}{E}\right) = \frac{1}{\sqrt{\pi}}A^{1/2}\frac{\sigma^2}{E} = \frac{a\sigma^2}{E}. \tag{3}$$

The stress intensity factor is then

$$K_{\mathrm{I}} = \sqrt{\mathcal{G}E} = \sqrt{a}\sigma.$$

The analytical solution is

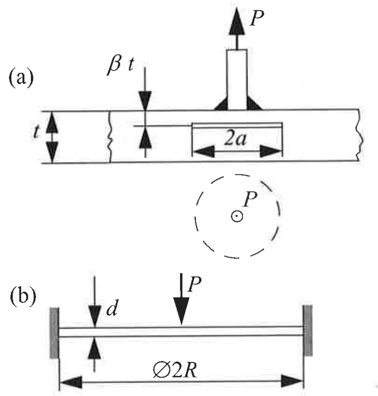
$$K_{\mathrm{I}} = \frac{2}{\sqrt{\pi}}\sqrt{a}\sigma.$$

Simple approach gives amazingly good result, the error is only -11 %.

Problem 3. A thick plate containg a circular delamination crack is loaded by a point force according to figure (a). Determine the stress intensity factor $K_{\rm I}$ and decide the critical load for fracture if $K_{\rm Ic} = 200$ MPam$^{1/2}$, $\beta = 0.1, t = 10\,{\rm cm}, a = 20\,{\rm cm}$.

Hint. For a circular rigidly fixed plate according to figure (b) the displacement of the loading point due to a force is

$$\Delta = \frac{3(1-\nu^2)}{4\pi}\frac{PR^2}{Ed^3}.$$



Solution. The potential energy at the solution point is

$$\Pi = -\frac{1}{2}P\delta = -\frac{1}{2}P^2\frac{3(1-\nu^2)R^2}{4\pi E(\beta t)^3}.$$

The crack driving force is thus

$$\mathcal{G} = -\frac{1}{2\pi a}\frac{{\rm d}\Pi}{{\rm d}a} = \frac{3P^2(1-\nu^2)}{8\pi^2 E(\beta t)^3}.$$

In plane strain we have

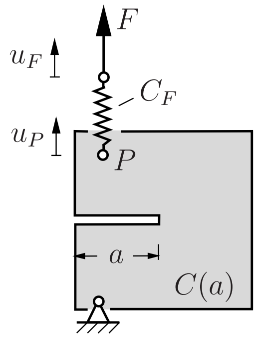
$$\mathcal{G} = \frac{K_{\rm I}^2}{E'} = \frac{K_{\rm I}(1-\nu^2)}{E}.$$

The design equation is to equate the stress intensity factor to the fracture toughness

$$K_{\rm I} = \frac{\sqrt{3}}{2\pi\sqrt{2}(\beta t)^{3/2}}P = K_{\rm Ic}.$$

The maximum allowable load is thus $P = 1.026$ MN.

Problem 4. Consider a DCB-test (Double Cantilever Beam) in a flexible testing apparatus. Draw the dimensionless crack driving force $\mathcal{G}/\mathcal{G}_c$ as a function of the dimensionless crack length a/a_0 using different values of the flexibility ratio $C_F/C(a_0)$. How flexible has the testing machine to be to produce unstable crack growth for a materal which has a shallow (nearly constant) R-curve?



Solution. The compliance function for the DCB-test is

$$C(a) = \frac{8}{EB}\left(\frac{a}{h}\right)^3, \quad \text{which has the value when } a = a_0 \text{ as} \quad C(a_0) = \frac{8}{EB}\left(\frac{a_0}{h}\right)^3.$$

Denoting the flexibility of the loading device as $C_F = \xi C(a_0)$, where ξ is a dimensionless parameter. The crack driving force is now

$$\mathcal{G} = \frac{u_F^2}{2}\frac{C'(a)}{(C(a)+C_F)^2} = \frac{u_F^2}{2}\frac{C'(a)}{C(a_0)^2[(a/a_0)^3+\xi]^2},$$

and has the treshold value

$$\mathcal{G}_c = \mathcal{G}(a_0) = \frac{u_F^2}{2}\frac{C'(a_0)}{(C(a_0)+C_F)^2} = \frac{u_F^2}{2}\frac{C'(a_0)}{C(a_0)^2(1+\xi)^2}.$$

Thus

$$\frac{\mathcal{G}}{\mathcal{G}_c} = \frac{C'(a)/C'(a_0)(1+\xi)^2}{[(a/a_0)^3+\xi]^2} = \frac{(a/a_0)^2(1+\xi)^2}{[(a/a_0)^3+\xi]^2}.$$

Let us investigate the behaviour of this expression when $a/a_0 \geq 1$, and denote $y = \mathcal{G}/\mathcal{G}_c$ and $x = a/a_0$, resulting in expression

$$y = \frac{(1+\xi)^2 x^2}{(x^3+\xi)^2}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^3+\xi)^2(1+\xi)^2\cdot 2x - (1+\xi)^2x^2\cdot 2(x^3+\xi)\cdot 3x^2}{(x^3+\xi)^2}.$$

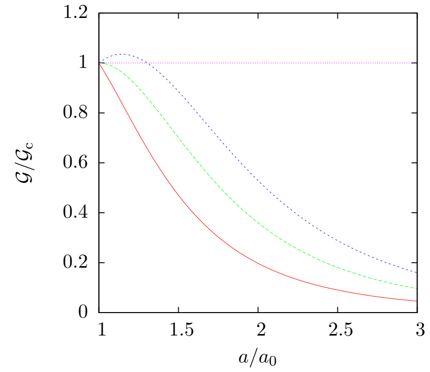
The numerator of the derivative is

$$(1+\xi)^2(x^3+\xi)[2x(\xi-2x^3)].$$

If the derivative expression is negative when $x > 1$ we have to have

$$\xi - 2x^3 < 0 \qquad \text{from which} \qquad \xi < 2x^3,$$

thus $\xi < 2$. The loading device should not be more flexible than $2C(a_0)$, if the R-curve is shallow. In the following figure the R-curve is drawn as a horizontal line. $\mathcal{G}/\mathcal{G}_c$-curves are drawn with the values of ξ as $\xi = 1, 2$ ja 3.


1.2
1
0.8
0.6
0.4
0.2
0
1
1.5
2
2.5
3
$\mathcal{G}/\mathcal{G}_c$
a/a_0