RAK-33060 Fracture mechanics and fatigue

1. Exercise

Problem 1. Consider the theoretical strength of a crystalline solid where the atoms are arranged into a regular cubic lattice with distance of d_0 between the neighbouring atom layers. The bonding force between two atoms can be obtained as $F = -\mathrm{d}\Psi/\mathrm{d}r$ where the following Lennart-Jones potential is adopted

$$\Psi = -A\left(\frac{d_0}{r}\right)^6 + B\left(\frac{d_0}{r}\right)^{12}.$$

The first terms represents attractive forces, while the second term describes repulsive ones. Determine the expression for the cohesive strength σ_c. The stress could be defined as

$$\sigma = -\frac{F}{d_0^2}.$$

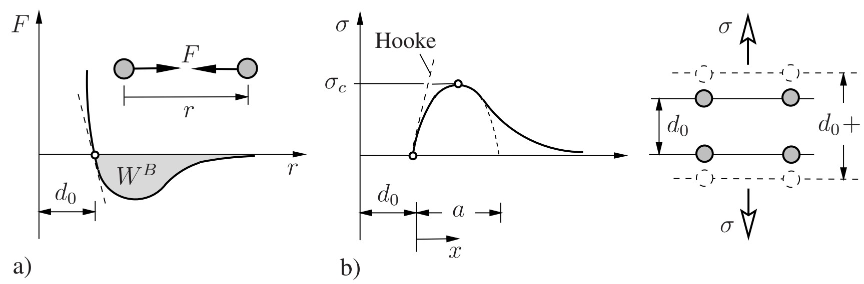
Strain ε can be defined naturally as

$$\varepsilon = \frac{x}{d_0} = \frac{r - d_0}{d_0},$$

where x is the distance from the equilibrium position. Determine also the expression for the surface energy γ_0

$$2\gamma_0 = \int_0^\infty \sigma \, \mathrm{d}x.$$

What are the values obtained for σ_c and γ_0 if the Young's modulus has the value $E = 210$ GPa and the distance between the atomic layers is $d_0 = 2.5 \cdot 10^{-10}$ m.



Solution. First we have to form the expression for the force. The unknown coefficients A and B can be solved from the following two conditions:

1. at equilibrium $r = d_0$, the force vanishes and
2. also the tangent of the stress-deformation relationship is the Young's modulus, i.e.

$$E = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}. \tag{1}$$

The expression for the force F is

$$F = -\frac{\mathrm{d}\psi}{\mathrm{d}r} = -6A\frac{d_0^6}{r^7} + 12B\frac{d_0^{12}}{r^{13}}.$$

From the equilibrium condition $F(d_0) = 0$ we get $A = 2B$. Expression for the stress is now

$$\sigma = -\frac{F}{d_0^2} = 12B\left(\frac{d_0^4}{r^7} - \frac{d_0^{10}}{r^{13}}\right).$$

Strain is

$$\varepsilon = \frac{r - d_0}{d_0}, \qquad \text{thus} \qquad \mathrm{d}\varepsilon = \frac{\mathrm{d}r}{d_0}.$$

The coefficient B is obtained from condition

$$\begin{aligned} E = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\mathrm{d}\sigma}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\varepsilon}\right|_{r=d_0} = d_0 \left.\frac{\mathrm{d}\sigma}{\mathrm{d}r}\right|_{r=d_0} \\ &= 12Bd_0 \left(-7\frac{d_0^4}{r^8} + 12\frac{d_0^{10}}{r^{14}}\right)\Bigg|_{r=d_0} = 72\frac{B}{d_0^3}, \end{aligned} \tag{2}$$

from which the solution is $B = Ed_0^3/72$. The necessary condition for the existence of a maximum in the stress-strain relationship is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} = 0, \qquad \text{which is equivalent to the condition} \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}r} = 0.$$

From equation (2) we get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}r} = -7\frac{d_0^4}{r^8} + 13\frac{d_0^{10}}{r^{14}} = 0, \qquad \text{from which} \qquad r_\mathrm{c} = (13/7)^{1/6}\, d_0.$$

Substituting this into the expression of the stress, we get

$$\sigma_\mathrm{c} = \frac{1}{6}E\left[(7/13)^{7/6} - (7/13)^{13/6}\right] \approx 0,037E.$$

The expression for the stress is thus

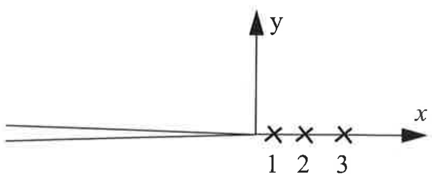
$$\sigma(r) = \frac{1}{6}E\left[\left(\frac{d_0}{r}\right)^7 - \left(\frac{d_0}{r}\right)^{13}\right].$$

The surface energy γ_0 is obtained by integrating the work done by the stress

$$\gamma_0 = \frac{1}{2}\int_0^\infty \sigma\,\mathrm{d}x = \frac{1}{2}\int_{d_0}^\infty \sigma\,\mathrm{d}r = \frac{E}{12}\Bigg|_{d_0}^{\infty}\left(-\frac{d_0^7}{6r^6} + \frac{d_0^{13}}{12r^{12}}\right) = \frac{1}{144}Ed_0 \approx 0,007Ed_0.$$

After substituting the values of E and d_0 we get the surface energy value 0.4 J/m^2. A typical value for iron is 2.2–2.8 J/m^2.

Problem 2. A two dimensional finite element analysis has been performed for a plate with a crack. The material is linearly isotropic elastic. In front of one of the crack tips the results shown in the table were obtained. Estimate the stress-intensity factors for this crack tip.



point	x [mm]	y [mm]	σ_x [MPa]	σ_y [MPa]	τ_{xy} [MPa]
1	0.10	0.0	1714.1	1712.1	1076.3
2	0.35	0.0	916.9	917.2	574.8
3	0.70	0.0	647.3	649.4	408.5

Solution. The K-dominated stress field is

$$\sigma_x = \sigma_y = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}, \quad \tau_{xy} = \frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}.$$

We can estimate the stress intensity factor by the least-squares method (LSM). The LSM error functions are

$$E_1(K_{\mathrm{I}}) = \frac{1}{2}\sum_{i=1}^{3}\left[\left(\frac{K_{\mathrm{I}}}{\sqrt{2\pi r_i}} - \sigma_x(r_i)\right)^2 + \left(\frac{K_{\mathrm{I}}}{\sqrt{2\pi r_i}} - \sigma_y(r_i)\right)^2\right]$$

and

$$E_2(K_{\mathrm{II}}) = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{K_{\mathrm{II}}}{\sqrt{2\pi r_i}} - \tau_{xy}(r_i)\right)^2.$$

For the existence of a minimum, the derivatives have to vanish, i.e.

$$\frac{\mathrm{d}E_1}{\mathrm{d}K_{\mathrm{I}}} = 0, \quad \text{and} \quad \frac{\mathrm{d}E_1}{\mathrm{d}K_{\mathrm{II}}} = 0.$$

This results in equations

$$\sum_{i=1}^{3}\frac{1}{\pi r_i}K_{\mathrm{I}} = \sum_{i=1}^{3}\frac{1}{\sqrt{2\pi r_i}}(\sigma_x(r_i) + \sigma_y(r_i)), \tag{3}$$

$$\sum_{i=1}^{3}\frac{1}{2\pi r_i}K_{\mathrm{II}} = \sum_{i=1}^{3}\frac{1}{\sqrt{2\pi r_i}}\tau_{xy}(r_i). \tag{4}$$

Substituting the values and solving we get the values $K_{\mathrm{I}} = 42.96$ MPa$\sqrt{\mathrm{m}}$ and $K_{\mathrm{II}} = 26.985$ MPa$\sqrt{\mathrm{m}}$.

The stress intensity factors can also be determined by extrapolation.

point	$K_{\mathrm{I}} = \sigma_y\sqrt{2\pi r}$ [MPa$\sqrt{\mathrm{m}}$]	$K_{\mathrm{I}} = \sigma_x\sqrt{2\pi r}$ [MPa$\sqrt{\mathrm{m}}$]	$K_{\mathrm{II}} = \tau_{xy}\sqrt{2\pi r}$ [MPa$\sqrt{\mathrm{m}}$]
1	42.92	42.97	26.97
2	43.01	42.99	26.96
3	43.06	42.92	27.09

Carry out the extrapolation to $r = 0$ and compare to the least-squares method!

Problem 3. In which direction θ is the largest shear stress found ar the tip of a crack loaded in mode III? The material is linear, isotropically elastic. Stress field in the vicinity of the crack tip is

$$\tau_{zx} = -\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\sin\tfrac{1}{2}\theta, \quad \tau_{zy} = -\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\cos\tfrac{1}{2}\theta,$$

and the other stress components vanish.

Solution. The resulting shear stress is

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \frac{1}{\sqrt{2\pi r}}K_{\mathrm{III}}, \quad \text{i.e. independent of the direction.}$$

Problem 4. Calculate and sketch the distribution of the Tresca effective stress around a crack tip loaded in mode I. The material is isotropic and linearly elastic with $\nu = 0.3$. The stress components in the Cartesian coordinate system are

$$\sigma_x = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \left[\cos\tfrac{1}{2}\theta\left(1 - \sin\tfrac{1}{2}\theta\sin\tfrac{3}{2}\theta\right)\right],$$
$$\sigma_y = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \left[\cos\tfrac{1}{2}\theta\left(1 + \sin\tfrac{1}{2}\theta\sin\tfrac{3}{2}\theta\right)\right],$$
$$\tau_{xy} = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \left(\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\cos\tfrac{3}{2}\theta\right), \quad \tau_{zy} = \tau_{zx} = 0.$$

Consider both plane stress $\sigma_z = 0$ and plane strain $\sigma_z = \nu(\sigma_x + \sigma_y)$.

The Tresca effective stress is

$$\sigma_\mathrm{e} = \max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|),$$

where σ_1, σ_2 and σ_3 are the principal stresses.

Solution. Both in plane stress and plane strain the stress in the z-direction (normal to the plane) is a principal stress (zero in plane stress). The in-plane principal stresses are obtained from the eigenvalue problem

$$\begin{pmatrix} \sigma_x - \sigma_i & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma_i \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which results in

$$\begin{aligned}
\sigma_i =& \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\
=& \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \left(\cos\tfrac{1}{2}\theta \pm \sqrt{(-\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\sin\tfrac{3}{2}\theta)^2 + (\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta\cos\tfrac{3}{2}\theta)^2}\right) \\
=& \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \left(1 \pm \sin\tfrac{1}{2}\theta\right)\cos\tfrac{1}{2}\theta.
\end{aligned}$$

Both the $\sigma_1 \geq \sigma_2 \geq 0$.

In plane stress the Tresca effective stress is thus

$$\sigma_\mathrm{e} = \sigma_1 = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \left(1 + \sin\tfrac{1}{2}\theta\right)\cos\tfrac{1}{2}\theta.$$

Denoting $f(\theta) = \left(1 + \sin\tfrac{1}{2}\theta\right)\cos\tfrac{1}{2}\theta$, the maximum is obtained when $f' = 0$, resulting in equation

$$\cos 2\theta = \sin\tfrac{1}{2}\theta,$$

which has a solution $\theta = \pi/3 = 60°$.