RAK-33060 Fracture mechanics and fatigue

1. Exercise

Problem 1. Consider the theoretical strength of a crystalline solid where the atoms are arranged into a regular cubic lattice with distance of \( d_0 \) between the neighbouring atom layers. The bonding force between two atoms can be obtained as \( F = -d\Psi/dr \) where the following Lennart-Jones potential is adopted

\[
\Psi = -A \left( \frac{d_0}{r} \right)^6 + B \left( \frac{d_0}{r} \right)^{12}.
\]

The first terms represents attractive forces, while the second term describes repulsive ones. Determine the expression for the cohesive strength \( \sigma_c \). The stress could be defined as

\[
\sigma = -\frac{F}{d_0^2}.
\]

Strain \( \varepsilon \) can be defined naturally as

\[
\varepsilon = \frac{x}{d_0} = \frac{r - d_0}{d_0},
\]

where \( x \) is the distance from the equilibrium position. Determine also the expression for the surface energy \( \gamma_0 \)

\[
2\gamma_0 = \int_0^\infty \sigma \, dx.
\]

What are the values obtained for \( \sigma_c \) and \( \gamma_0 \) if the Young’s modulus has the value \( E = 210 \) GPa and the distance between the atomic layers is \( d_0 = 2.5 \cdot 10^{-10} \) m.

Solution. First we have to form the expression for the force. The unknown coefficients \( A \) and \( B \) can be solved from the following two conditions:

1. at equilibrium \( r = d_0 \), the force vanishes and
2. also the tangent of the stress-deformation relationship is the Young’s modulus, i.e.

\[
E = \frac{d\sigma}{d\varepsilon} \bigg|_{\varepsilon=0}.
\]

The expression for the force \( F \) is

\[
F = -\frac{d\psi}{dr} = -6A \frac{d_0^6}{r^7} + 12B \frac{d_0^{12}}{r^{13}}.
\]

From the equilibrium condition \( F(d_0) = 0 \) we get \( A = 2B \). Expression for the stress is now

\[
\sigma = -\frac{F}{d_0^2} = 12B \left( \frac{d_0^4}{r^7} - \frac{d_0^{10}}{r^{13}} \right).
\]
Strain is

\[ \varepsilon = \frac{r - d_0}{d_0}, \quad \text{thus} \quad d\varepsilon = \frac{dr}{d_0}. \]

The coefficient \( B \) is obtained from condition

\[ E = \frac{d\sigma}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{d\sigma}{dr} \frac{dr}{d\varepsilon} \bigg|_{r=d_0} = d_0 \frac{d\sigma}{dr} \bigg|_{r=d_0} \]

\[ = 12Bd_0 \left( -7\frac{d_0^6}{r^8} + 12\frac{d_0^{10}}{r^{14}} \right) \bigg|_{r=d_0} = 72 \frac{B}{d_0^3}, \]

from which the solution is \( B = Ed_0^3/72 \). The necessary condition for the existence of a maximum in the stress-strain relationship is

\[ \frac{d\sigma}{d\varepsilon} = 0, \quad \text{which is equivalent to the condition} \quad \frac{d\sigma}{dr} = 0. \]

From equation (2) we get

\[ \frac{d\sigma}{dr} = -7\frac{d_0^6}{r^8} + 13\frac{d_0^{10}}{r^{14}} = 0, \quad \text{from which} \quad r_c = (13/7)^{1/6} d_0. \]

Substituting this into the expression of the stress, we get

\[ \sigma_c = \frac{1}{6} E \left[ (\frac{7}{13})^{7/6} - (\frac{7}{13})^{13/6} \right] \approx 0.037 E. \]

The expression for the stress is thus

\[ \sigma(r) = \frac{1}{6} E \left[ \left( \frac{d_0}{r} \right)^7 - \left( \frac{d_0}{r} \right)^{13} \right]. \]

The surface energy \( \gamma_0 \) is obtained by integrating the work done by the stress

\[ \gamma_0 = \frac{1}{2} \int_0^\infty \sigma \, dx = \frac{1}{2} \int_0^d \sigma \, dr = \frac{E}{12} \left|_{d_0}^\infty \right. \left( -\frac{d_0^7}{6r^6} + \frac{d_0^{13}}{12r^{12}} \right) = \frac{1}{144} Ed_0 \approx 0.007 Ed_0. \]

After substituting the values of \( E \) and \( d_0 \) we get the surface energy value 0.4 J/m². A typical value for iron is 2.2–2.8 J/m².

**Problem 2.** A two dimensional finite element analysis has been performed for a plate with a crack. The material is linearly isotropic elastic. In front of one of the crack tips the results shown in the table were obtained. Estimate the stress-intensity factors for this crack tip.

<table>
<thead>
<tr>
<th>point</th>
<th>( x ) [mm]</th>
<th>( y ) [mm]</th>
<th>( \sigma_x ) [MPa]</th>
<th>( \sigma_y ) [MPa]</th>
<th>( \tau_{xy} ) [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>0.0</td>
<td>1714.1</td>
<td>1712.1</td>
<td>1076.3</td>
</tr>
<tr>
<td>2</td>
<td>0.35</td>
<td>0.0</td>
<td>916.9</td>
<td>917.2</td>
<td>574.8</td>
</tr>
<tr>
<td>3</td>
<td>0.70</td>
<td>0.0</td>
<td>647.3</td>
<td>649.4</td>
<td>408.5</td>
</tr>
</tbody>
</table>
Solution. The $K$-dominated stress field is

$$\sigma_x = \sigma_y = \frac{K_I}{\sqrt{2\pi r}}, \quad \tau_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}.$$ 

We can estimate the stress intensity factor by the least-squares method (LSM). The LSM error functions are

$$E_1(K_I) = \frac{1}{2} \sum_{i=1}^{3} \left[ \left( \frac{K_I}{\sqrt{2\pi r_i}} - \sigma_x(r_i) \right)^2 + \left( \frac{K_I}{\sqrt{2\pi r_i}} - \sigma_y(r_i) \right)^2 \right]$$

and

$$E_2(K_{II}) = \frac{1}{2} \sum_{i=1}^{3} \left( \frac{K_{II}}{\sqrt{2\pi r_i}} - \tau_{xy}(r_i) \right)^2.$$ 

For the existence of a minimum, the derivatives have to vanish, i.e.

$$\frac{dE_1}{dK_I} = 0, \quad \text{and} \quad \frac{dE_1}{dK_{II}} = 0.$$ 

This results in equations

$$\sum_{i=1}^{3} \frac{1}{\pi r_i} K_I = \sum_{i=1}^{3} \frac{1}{\sqrt{2\pi r_i}} (\sigma_x(r_i) + \sigma_y(r_i)), \quad (3)$$

and

$$\sum_{i=1}^{3} \frac{1}{2\pi r_i} K_{II} = \sum_{i=1}^{3} \frac{1}{\sqrt{2\pi r_i}} \tau_{xy}(r_i). \quad (4)$$

Substituting the values and solving we get the values $K_I = 42.96 \text{ MPa}\sqrt{\text{m}}$ and $K_{II} = 26.985 \text{ MPa}\sqrt{\text{m}}$.

The stress intensity factors can also be determined by extrapolation.

<table>
<thead>
<tr>
<th>point</th>
<th>$K_I = \sigma_y \sqrt{2\pi r} \text{ [MPa}\sqrt{\text{m]}$</th>
<th>$K_I = \sigma_x \sqrt{2\pi r} \text{ [MPa}\sqrt{\text{m]}$</th>
<th>$K_{II} = \tau_{xy} \sqrt{2\pi r} \text{ [MPa}\sqrt{\text{m]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>42.92</td>
<td>42.97</td>
<td>26.97</td>
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<tr>
<td>2</td>
<td>43.01</td>
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</tr>
<tr>
<td>3</td>
<td>43.06</td>
<td>42.92</td>
<td>27.09</td>
</tr>
</tbody>
</table>

Carry out the extrapolation to $r = 0$ and compare to the least-squares method!

Problem 3. In which direction $\theta$ is the largest shear stress found at the tip of a crack loaded in mode III? The material is linear, isotropically elastic. Stress field in the vicinity of the crack tip is

$$\tau_{zx} = \frac{-K_{III}}{\sqrt{2\pi r}} \sin \frac{1}{2} \theta, \quad \tau_{zy} = \frac{-K_{III}}{\sqrt{2\pi r}} \cos \frac{1}{2} \theta,$$

and the other stress components vanish.

Solution. The resulting shear stress is

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \frac{1}{\sqrt{2\pi r}} K_{III}, \quad \text{i.e. independent of the direction.}$$
Problem 4. Calculate and sketch the distribution of the Tresca effective stress around a crack tip loaded in mode I. The material is isotropic and linearly elastic with $\nu = 0.3$. The stress components in the Cartesian coordinate system are

$$\sigma_x = \frac{K_i}{\sqrt{2\pi r}} \left[ \cos \frac{1}{2} \theta \left( 1 - \sin \frac{1}{2} \theta \sin \frac{3}{2} \theta \right) \right],$$

$$\sigma_y = \frac{K_i}{\sqrt{2\pi r}} \left[ \cos \frac{1}{2} \theta \left( 1 + \sin \frac{1}{2} \theta \sin \frac{3}{2} \theta \right) \right],$$

$$\tau_{xy} = \frac{K_i}{\sqrt{2\pi r}} \left( \cos \frac{1}{2} \theta \sin \frac{1}{2} \theta \cos \frac{3}{2} \theta \right), \quad \tau_{yz} = \tau_{zx} = 0.$$

Consider both plane stress $\sigma_z = 0$ and plane strain $\sigma_z = \nu (\sigma_x + \sigma_y)$.

The Tresca effective stress is

$$\sigma_e = \max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|),$$

where $\sigma_1, \sigma_2$ and $\sigma_3$ are the principal stresses.

Solution. Both in plane stress and plane strain the stress in the $z$-direction (normal to the plane) is a principal stress (zero in plane stress). The in-plane principal stresses are obtained from the eigenvalue problem

$$\begin{pmatrix} \sigma_x - \sigma_i & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma_i \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which results in

$$\sigma_i = \frac{1}{2} (\sigma_x + \sigma_y) \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$= \frac{K_i}{\sqrt{2\pi r}} \left( \cos \frac{1}{2} \theta \pm \sqrt{(- \cos \frac{1}{2} \theta \sin \frac{1}{2} \theta \sin \frac{3}{2} \theta)^2 + (\cos \frac{1}{2} \theta \sin \frac{1}{2} \theta \cos \frac{3}{2} \theta)^2} \right)$$

$$= \frac{K_i}{\sqrt{2\pi r}} (1 \pm \sin \frac{1}{2} \theta) \cos \frac{1}{2} \theta.$$

Both the $\sigma_1 \geq \sigma_2 \geq 0$.

In plane stress the Tresca effective stress is thus

$$\sigma_e = \sigma_1 = \frac{K_i}{\sqrt{2\pi r}} (1 + \sin \frac{1}{2} \theta) \cos \frac{1}{2} \theta.$$

Denoting $f(\theta) = (1 + \sin \frac{1}{2} \theta) \cos \frac{1}{2} \theta$, the maximum is obtained when $f' = 0$, resulting in equation

$$\cos 2\theta = \sin \frac{1}{2} \theta,$$

which has a solution $\theta = \pi/3 = 60^\circ$. 