Continuum mechanics - 2020

1. home exercise – mathematical preliminaries

Problem 1 corrected 4.9.2020. The tensor $A$ should be symmetric.

1. Let a tensor be given as

$$A = \alpha (I - e_1 \otimes e_1) + \beta (e_1 \otimes e_2 + e_2 \otimes e_1),$$

where $\alpha, \beta$ are scalars and $e_1, e_2$ orthogonal unit vectors.

(a) Show that the eigenvalues $\lambda_i, i = 1, 2, 3$, of $A$ are

$$\lambda_1 = \alpha, \quad \lambda_2 = \frac{1}{2} \alpha + \sqrt{\frac{1}{4} \alpha^2 + \beta^2}, \quad \lambda_3 = \frac{1}{2} \alpha - \sqrt{\frac{1}{4} \alpha^2 + \beta^2}.$$

(b) Derive the associated normalized eigenvectors $\hat{n}_i, i = 1, 2, 3$, which in matrix representation are

$$[\hat{n}_1] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\hat{n}_2] = \frac{1}{\sqrt{1 + (\lambda_2/\beta)^2}} \begin{bmatrix} 1 \\ \lambda_2/\beta \\ 0 \end{bmatrix}, \quad [\hat{n}_3] = \frac{1}{\sqrt{1 + (\lambda_3/\beta)^2}} \begin{bmatrix} 1 \\ \lambda_3/\beta \\ 0 \end{bmatrix}.$$

2. Show that the invariants of the deviatoric part of the tensor $A$, $J_A^2 = -\frac{1}{2} \text{tr} (\text{dev} A)^2$, $J_A^3 = \det (\text{dev} A) = \frac{1}{3} \text{tr} (\text{dev} A)^3$ can be expressed in terms of the invariants of $A$, $I_A^1, I_A^2$ and $I_A^3$ as follows

$$J_A^2 = I_A^2 - \frac{1}{3} (I_A^1)^2,$$

$$J_A^3 = I_A^3 - \frac{1}{3} I_A^1 I_A^2 + \frac{2}{27} (I_A^1)^3.$$

Hint: Make use of the definitions

$$I_A^1 = \text{tr} A, \quad I_A^2 = \frac{1}{2} [\text{tr} (A^2) - \text{tr}(A^2)]$$

and the Cayley-Hamilton equation (tensor itself satisfies its characteristic equation)

$$A^3 - I_A^1 A^2 + I_A^2 A - I_A^3 I = 0.$$

3. Show that

$$\frac{\partial \det A}{\partial A} = (A^T)^2 - I_A^1 A^T + I_A^2 I.$$

Hint: Derive first the results

$$\frac{\partial \text{tr} A}{\partial A} = I, \quad \frac{\partial \text{tr} A^2}{\partial A} = 2A^T, \quad \frac{\partial \text{tr} A^3}{\partial A} = 3(A^T)^2.$$

Make use of the Cayley-Hamilton equation by takin its trace and derive it with respect to tensor $A$.

4. Let a tensor valued tensor function be given as

$$f(A) = \alpha_6 A^6 + \alpha_4 A^4,$$

where $\alpha_4, \alpha_6$ are scalars and $A$ is a symmetric second order tensor (in a 3-D space). Derive its expression given with tensor powers at most of second order, i.e.

$$f(A) = \beta_2 A^2 + \beta_1 A + \beta_0 I.$$

Return in the Moodle system at latest on Tuesday 15.9.2020.