Continuum mechanics - 2018

2. home exercise – kinematics, stress

1. A motion is given in the material description (Lagrangian formulation) as

\[ \begin{align*}
    x_1 &= X_1 \exp(t/t_0) + X_3[\exp(t/t_0) - 1], \\
    x_2 &= X_2 + X_3[\exp(t/t_0) - \exp(-t/t_0)], \\
    x_3 &= X_3.
\end{align*} \]

(a) Find out the motion in the spatial description (Eulerian formulation).
(b) Determine the velocity- and acceleration fields both in the material and spatial description.

2. A spherical cavity of radius \( A \) at time \( t = 0 \) in an infinite body is centered at the origin. An explosion inside the cavity at \( t = 0 \) produces the motion

\[ \mathbf{x} = \frac{f(R, t)}{R} \mathbf{X}, \] (1)

where \( R = |\mathbf{X}| = \sqrt{X^T \mathbf{X}} \) is the magnitude of the position vector in the reference configuration. The cavity wall has a radial motion given in Eqn. (1) such that at time \( t \) the cavity is spherical with radius \( a(t) \).

(a) Find the deformation gradient, \( \mathbf{F} \), and the Jacobian of the transformation, \( J \).
(b) Find the velocity and acceleration fields.
(c) Show that if the motion is restricted to be isochoric, then

\[ f(R, t) = (R^3 + a^3 - A^3)^{1/3}. \]

3. Given the velocity field

\[ \begin{align*}
    v_1 &= \exp(x_3 - ct) \cos \omega t, \\
    v_2 &= \exp(x_3 - ct) \sin \omega t, \\
    v_3 &= c = \text{constant}.
\end{align*} \]

(a) Show that the speed (magnitude of the velocity) of every particle is constant.
(b) Calculate the acceleration components \( a_i \) (Note that the previous part implies that \( a_i v_i = 0 \).
(c) Find the logarithmic rate of stretching, \( \dot{\alpha}/\alpha \), for a line element that is in the direction of \( (1/\sqrt{2}, 0, 1/\sqrt{2}) \) in the deformed configuration at \( \mathbf{x} = \mathbf{0} \).
(d) Integrate the velocity equation to find the motion \( \mathbf{x} = \varphi(\mathbf{X}, t) \) using the initial conditions that at \( t = 0, \mathbf{x} = \mathbf{X} \). Hint: Integrate the \( v_3 \) equation first.

4. Investigate a simple shear which depends linearly on time \( t \) as

\[ \begin{align*}
    x_1 &= X_1 + (t/t_0)X_2, \\
    x_2 &= X_2, \\
    x_3 &= X_3.
\end{align*} \]

Assuming a hypoelastic material model

\[ \dot{\sigma} = c : \mathbf{d}, \]
where $\sigma$ is the Jaumann-Zaremba stress rate$^1$

$$
\dot{\sigma} = \sigma - w\sigma - \sigma w^T,
$$

where $w$ is the spin tensor, i.e. the skew-symmetric part of the spatial velocity gradient $l$. For the spatial elasticity tensor $c$ the following form is assumed

$$
c_{ijkl} = G \left( \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl} \right).
$$

Determine the expressions for the stress components $\sigma_{11}, \sigma_{12}$ and $\sigma_{22}$ and draw the results in time range $0 \leq t \leq 5t_0$. The initial state can be assumed to be stress free.

Solve the problem also if the Jaumann-Zaremba rate is replaced by the Oldroyd rate in the hypoelastic model

$$
\nabla \sigma = c \cdot d, \quad \nabla \sigma = \dot{\sigma} - l\sigma - \sigma l^T.
$$

Notice that there is no difference between the Oldroyd and the Truesdell rate in this case. Why?

What are your conclusions?

5. For an elastic isotropic solid the specific strain energy $W = \rho_0 \psi$ on is expressed in terms of the invariants of the right Cauchy-Green deformation tensor

$$
I_1 = \text{tr} \, C, \quad I_2 = \frac{1}{2}(\text{tr} \,(C^2) - I_1^2), \quad I_3 = \text{det} \, C = \frac{1}{3}(\text{tr} \,(C^3) - 3I_2I_1 - I_1^3).
$$

Show that the second Piola-Kirchhoff stress tensor has the expression

$$
S = \frac{\partial W}{\partial E} = 2\frac{\partial W}{\partial C} = 2 \left[ \left( \frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) I - \frac{\partial W}{\partial I_2} C + J^2 \frac{\partial W}{\partial I_3} C^{-1} \right],
$$

where $J = \text{det} \, F$.

6. (Holzapfel problem 4b, on page 250) The so-called Saint-Venant Kirchhoff material model is characterized by the specific strain energy function

$$
W = \frac{1}{2}\lambda(\text{tr} \, E)^2 + \mu \text{tr} \,(E^2),
$$

where $\lambda$ and $\mu$ are the Lamé’s constants.

(a) Derive the stress-strain relationship between the 2nd Piola-Kirchhoff stress and the Green-Lagrange strain tensors.

(b) Consider a uniaxial homogeneous deformation of a bar with a uniform cross-section, i.e. $x = \alpha X$. Derive the relationship between the PK1-stress $P$ and the stretch ratio $\alpha$ (which is a cubic equation in $\alpha$). Draw the picture.

(c) Show that $P(\alpha)$ is not monotonic in compression and derive the critical stretch ratio $\alpha_{cr} = (1/3)^{1/2}$, at which the Saint-Venant Kirchhoff model fails i.e. zero stiffness, the tangent of $P(\alpha)$ at $\alpha_{cr}$. This failure is not influenced by the material constants $\lambda, \mu$.

$^1$The hypoelastic models with the Jaumann-Zaremba stress rate are frequently used in many commercial FE-programs.
(d) In addition show that the stress tends to zero in the compressive limit, i.e. \( \alpha \to 0^+ \), which is physically unrealistic.

7. The specific strain energy for the modified Saint-Venant Kirchhoff material model is

\[
W = \frac{1}{2} \kappa (\ln J)^2 + \mu \text{tr} (E^2),
\]

where \( J = \det \mathbf{F} \), \( \mu > 0 \) the shear modulus and \( \kappa \) the compressibility modulus.

(a) Derive the constitutive equation between the PK2-stress tensor \( \mathbf{S} \) and the right Cauchy-Green deformation tensor \( \mathbf{C} \).

(b) As in the previous problem, consider a uniform bar in a uniaxial homogeneous stress state and determine the relationship between the nominal stress \( P \) and the stretch ratio \( \alpha \). Investigate the behaviour of the constitutive function in compression \( 0 < \alpha < 1 \) and in tension \( \alpha > 1 \) and show that in compression \( P \to -\infty \) when \( \alpha \to 0^+ \).

8. Show that

\[
\sigma_{11} = 2(\alpha_1^2 - \alpha_1^{-2} \alpha_2^{-2}) \frac{\partial \psi}{\partial I_1} - 2(\alpha_1^{-2} - \alpha_1^2 \alpha_2^2) \frac{\partial \psi}{\partial I_2},
\]

\[
\sigma_{22} = 2(\alpha_2^2 - \alpha_1^{-2} \alpha_2^{-2}) \frac{\partial \psi}{\partial I_1} - 2(\alpha_2^{-2} - \alpha_1^2 \alpha_2^2) \frac{\partial \psi}{\partial I_2},
\]

are the nonzero Cauchy stress components for biaxial tensile loading of a homogeneous, isotropic, incompressible rubber-like material by solving for the indeterminant pressure \( p \) from the \( \sigma_{33} = 0 \) condition. **Hint.** First express the invariants of the left Cauchy-Green deformation tensor \( I_1 \) and \( I_2 \) in terms of the principal stretches \( \alpha_i \) and take the incompressibility condition into account. The specific Helmholtz free energy is now defined per unit reference volume.

9. A unit cube of incompressible isotropic elastic material undergoes the finite deformation

\[
x_1 = \alpha X_1, \quad x_2 = \alpha^{-1} X_2, \quad x_3 = X_3,
\]

where \( \alpha \) is constant. The specific Helmholtz free energy per unit reference volume is

\[
\psi = C_1(I_1 - 3) + C_2(I_2 - 3) - \frac{1}{2} p(I_3 - 1).
\]

where \( C_1 \) and \( C_2 \) are material constants and \( I_i \) are the principal invariants of the left Cauchy-Green deformation tensor (Mooney-Rivlin material model). Draw the deformed cube, noting the lengths of its edges. Find the stress and hence determine the total loads \( F_1, F_2 \) and \( F_3 \) acting on the faces normal to the \( X_1, X_2 \) and \( X_3 \) directions. Show that when \( C_1 > 3C_2 \), there are three values of \( \alpha \) for which the body is in equilibrium with \( F_1 = F_2 = F_3 \), and find these values.

10. The following table shows the results from a uniaxial test (compression/tension). The table shows the stretch \( \alpha_1 \) and the nominal stress \( P_{11} \) (stress with respect to the undeformed cross-sectional area).
(a) Calibrate the neo-Hooke, Mooney-Rivlin and Yeoh model parameters using the least-squares method. Plot and comment the results. Use the following error function

\[ E = \frac{1}{2} \sum_{i=1}^{25} \left( \frac{P_{\text{mall}}^{11}(\alpha_i)}{P_{\text{kie}}^{11}(\alpha_i)} - 1 \right)^2. \]

(b) Determine also the response of these models in simple shear. Plot the results.

The Mooney-Rivlin model has two parameters

\[ \psi = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) - \frac{1}{2} p(I_3 - 1). \]

Yeoh noticed in 1990 that the second invariant has little influence on the response and formulated a three parameter version based on only the first invariant

\[ \psi = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3 - \frac{1}{2} p(I_3 - 1). \]

Last day to return is Friday 11.1.2019.