Continuum mechanics - 2018

1. home exercise – mathematical preliminaries, balance equations

1. Show that the invariants of the deviatoric part of the tensor $\mathbf{A}$, $J_2^A, J_3^A$ can be expressed in terms of the invariants of $\mathbf{A}$, $I_1^A, I_2^A$ and $I_3^A$ as follows

\[
J_2^A = I_2^A - \frac{1}{3}(I_1^A)^2,
J_3^A = I_3^A - \frac{1}{3}I_1^A I_2^A + \frac{2}{27}(I_1^A)^3.
\]

Hint: Make use of the definitions

\[
I_1^A = \text{tr} \mathbf{A}, \quad I_2^A = \frac{1}{2}[(\text{tr} \mathbf{A})^2 - \text{tr}(\mathbf{A}^2)]
\]

and the Cayley-Hamilton equation (tensor itself satisfies its characteristic equation)

\[
\mathbf{A}^3 - I_1^A \mathbf{A}^2 + I_2^A \mathbf{A} - I_3^A \mathbf{I} = 0.
\]

2. Suppose that $\mathbf{Q} = \mathbf{Q}(t)$ is a given orthogonal tensor-valued function of a scalar, such as time $t$. Show that $\dot{\mathbf{Q}} \mathbf{Q}^T$ is a skew tensor whith the property $\dot{\mathbf{Q}} \mathbf{Q}^T = -(\dot{\mathbf{Q}} \mathbf{Q}^T)^T$.

3. Show that

\[
\frac{\partial \det \mathbf{A}}{\partial \mathbf{A}} = (\mathbf{A}^T)^2 - I_1^A \mathbf{A}^T + I_2^A \mathbf{I}.
\]

Hint: Derive first the results

\[
\frac{\partial \text{tr} \mathbf{A}}{\partial \mathbf{A}} = \mathbf{I}, \quad \frac{\partial \text{tr} \mathbf{A}^2}{\partial \mathbf{A}} = 2\mathbf{A}^T, \quad \frac{\partial \text{tr} \mathbf{A}^3}{\partial \mathbf{A}} = 3(\mathbf{A}^T)^2.
\]

Make use of the Cayley-Hamilton equation by takin its trace and derive it with respect to tensor $\mathbf{A}$.

4. A velocity field of a plane motion has components of the form

\[
v_1 = (\alpha x_1 - \beta x_2)t, \quad v_2 = \beta x_1 - \alpha x_2, \quad v_3 = 0,
\]

where $\alpha$ and $\beta$ are positive constants. Assume that the spatial mass density $\rho$ is independent of the current position $\mathbf{x}$ so that $\text{grad} \rho = 0$.

(a) Express $\rho$ so that the continuity mass equation is satisfied.

(b) Find a condition for which the given motion is isochoric.

5. Two motions of a continuum body are given in the form

\[
\mathbf{x} = (1 + \alpha(t)t) \mathbf{X}, \quad \mathbf{x} = \mathbf{X} + \alpha(t)t^2 (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) \mathbf{X},
\]

with the scalar function $\alpha(t)$ and the set $\{\mathbf{e}_a\}, a = 1, 2, 3$, of orthogonal unit vectors.

Find expression for the spatial mass density $\rho$ in terms of $\rho_0$ so that the continuity mass equation is satisfied.
6. The Cauchy stress distribution of a continuum is expressed with respect to a rectangular $x_1, x_2, x_3$-coordinate system. The matrix representation of the Cauchy stress tensor is given in the form

$$\sigma = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & x_2 + \alpha x_3 & \Phi(x_2, x_3) \\ 0 & \Phi(x_2, x_3) & x_2 + \beta x_3 \end{bmatrix},$$

where $\alpha$ and $\beta$ are scalar constants.

(a) Find $\Phi(x_2, x_3)$ so that the given spatial stress field satisfies $\text{div}\sigma = 0$.

(b) Consider $\Phi(x_2, x_3)$ from (a) and find the Cauchy traction vector $t$ on the plane $\Psi = x_1 + x_2 + x_3 = 0$.

Last day to return is Friday 16.11.2018.