Continuum mechanics

8. exercise – continuum thermodynamics, Legendre transformation

1. Derive the local energy equation for a heat conducting magneto-electro-mechanical body. Start with the global energy equation

\[ \dot{K} + \dot{U} = P_{\text{mech}} + P_{\text{heat}} + P_{\text{em}}, \]  

(1)

where \( K \) is the kinetic energy, \( U \) the internal energy, \( P_{\text{mech}} \), \( P_{\text{heat}} \) and \( P_{\text{em}} \) are the powers of mechanical, heat and electro-magnetic energy inputs, respectively. The electro-magnetic energy input to the system can be assumed to be

\[ P_{\text{em}} = - \oint_{\partial V} E \times H \, dV, \]

where \( E \) and \( H \) are the electric and magnetic field strengths, respectively. The Faraday and Ampère laws in the global form are

\[ \oint_{\partial A} E \cdot ds = - \frac{\partial}{\partial t} \int_A B \cdot dA, \]  

(2)

\[ \oint_{\partial A} H \cdot ds = \frac{\partial}{\partial t} \int_A D \cdot dA + \int_A J \cdot dA, \]  

(3)

and the corresponding local forms are

\[ \text{curl} \, E = - \frac{\partial B}{\partial t}, \]  

(4)

\[ \text{curl} \, H = J + \frac{\partial D}{\partial t}, \]  

(5)

where \( B \) is the magnetic flux density (or magnetic induction), \( J \) is the current density, \( D \) the electric flux density and \( dA \) is differential oriented area, \( ds \) differential line element.

Derive also the Clausius-Duhem inequality.

Solution:

\[ K = \int \frac{1}{2} v^2 \, dm = \int \frac{1}{2} \rho v^2 \, dV \quad \dot{K} = \int \rho \dot{v} \, dV, \]

\[ U = \int edm = \int \rho e \, dV \quad \dot{U} = \int \rho \dot{e} \, dV \]

\[ P_{\text{mech}} = P_{\text{surface traction}} + P_{\text{Body Force}} = \int v \cdot t \, dA + \int v \cdot b \, dV \]

\[ P_{\text{heat}} = P_{\text{Internal heat}} - P_{\text{Surface heat}} = \int \rho r \, dV - \oint q \cdot n \, dA \]

\[ P_{\text{em}} = - \oint_{\partial V} E \times H \, dV, \]

\footnote{In the study book \( P_{\text{mech}} \) and \( P_{\text{heat}} \) are denoted as \( P_{\text{ext}} \) and \( R \).}
According to the divergence theorem:
\[ \oint_S (F \cdot n) dS = \int_V (\text{div} F) dV \]

Therefore by replacing above values in equation (1):
\[ \int [\rho v \dot{v} + \rho \dot{e} - \text{div} \sigma - \rho b v - \rho r + \text{div} q + \text{div}(E \times H)] dV = 0 \]

(Note1: \( \text{div} \sigma = v \text{div} \sigma + \text{grad} : \sigma \))
\[ \int [u.(\rho v - \text{div} \sigma - \rho b) + \rho \dot{e} - \text{grad} : \sigma - \rho r + \text{div} q + \text{div}(E \times H)] dV = 0 \]

(Note2: \( \rho v - \text{div} \sigma - \rho b = 0 \))
(Note3: \( \text{div}(E \times H) = H \text{curl} E - E \text{curl} H = H (-\dot{B}) - E (J + \dot{D}) \))

Therefore, first thermodynamic law:
\[ \rho \dot{e} = \text{grad} : \sigma + \rho r - \text{div} q + E (J + \dot{D}) + H \cdot \dot{B} \]

Second thermodynamic law:
\[ \dot{S} \geq \int \frac{\rho r}{\theta} dV - \oint \frac{q \cdot n}{\theta} dA \]

Therefore:
\[ \int \rho \dot{s} dV - \int \frac{\rho r}{\theta} dV + \int \text{div} \left( \frac{q}{\theta} \right) dV \geq 0 \]
\[ = \int \left( \rho \dot{s} - \frac{\rho r}{\theta} + \frac{1}{\theta^2} \text{div} q - \frac{1}{\theta^2} q \cdot \text{grad} \theta \right) dV \geq 0 \]

Then:
\[ \rho \dot{s} - \frac{1}{\theta} (\rho r - \text{div} q) - \frac{1}{\theta^2} q \cdot \text{grad} \theta \geq 0 \]

2. Determine the Legendre-Fenchel dual funktion \( g(y) \) for the following functions:
   (a) \( f(x) = \frac{1}{4} x^4 \),
   (b) \( f(x) = k |x| \).

Show that the dual functions \( g(y) \) are convex. The Legendre-Fenchel dual function is defined as
\[ g(y) = \sup_{x \in \mathbb{R}} [yx - f(x)] . \]

**Solution:**
(a) 
\[ f(x) = \frac{1}{4} x^4 \rightarrow \frac{\partial (xy - \frac{1}{4} x^4)}{\partial x} = 0 \rightarrow y = x^3 \rightarrow x = \sqrt[3]{y} \]
Thus:

\[ g(y) = \frac{3}{4} y^\frac{4}{3} \]

In order to \( g(y) \) be convex; \( g''(y) \geq 0 \)

\[ g''(y) = \frac{1}{3\sqrt[3]{y^2}} > 0 \]

(b)
We can consider this \( f(x) \) function in two different cases:
If \( x > 0 \):

\[ f(x) = kx \rightarrow \frac{\partial(xy - kx)}{\partial x} = 0 \rightarrow y = k \]

\[ g(y) = kx - kx = 0 \rightarrow g''(y) \geq 0 \]

If \( x < 0 \):

\[ f(x) = -kx \rightarrow \frac{\partial(xy + kx)}{\partial x} = 0 \rightarrow y = -k \]

\[ g(y) = -kx + kx = 0 \rightarrow g''(y) \geq 0 \]