Continuum mechanics

8. exercise – continuum thermodynamics, Legendre transformation

1. Derive the local energy equation for a heat conducting magneto-electro-mechanical body. Start with the global energy equation

\[ \dot{\mathcal{K}} + \dot{\mathcal{U}} = \mathcal{P}^{\text{mech}} + \mathcal{P}^{\text{heat}} + \mathcal{P}^{\text{em}}, \]  

(1)

where \( \mathcal{K} \) is the kinetic energy, \( \mathcal{U} \) the internal energy, \( \mathcal{P}^{\text{mech}} \), \( \mathcal{P}^{\text{heat}} \) and \( \mathcal{P}^{\text{em}} \) are the powers of mechanical, heat and electro-magnetic energy inputs, respectively\(^1\). The electro-magnetic energy input to the system can be assumed to be

\[ \mathcal{P}^{\text{em}} = -\int_{\partial V} E \times H \, dV, \]

where \( E \) and \( H \) are the electric and magnetic field strengths, respectively. The Faraday and Ampère laws in the global form are

\[ \oint_{\partial A} E \cdot ds = -\frac{\partial}{\partial t} \int_{A} B \cdot dA, \]  

(2)

\[ \oint_{\partial A} H \cdot ds = \frac{\partial}{\partial t} \int_{A} D \cdot dA + \int_{A} J \cdot dA, \]  

(3)

and the corresponding local forms are

\[ \text{curl } E = -\frac{\partial B}{\partial t}, \]  

(4)

\[ \text{curl } H = J + \frac{\partial D}{\partial t}, \]  

(5)

where \( B \) is the magnetic flux density (or magnetic induction), \( J \) is the current density, \( D \) the electric flux density and \( dA \) is differential oriented area, \( ds \) differential line element.

Derive also the Clausius-Duhem inequality.

2. Determine the Legendre-Fenchel dual function \( g(y) \) for the following functions:

(a) \( f(x) = \frac{1}{4} x^4 \),

(b) \( f(x) = k|x| \).

Show that the dual functions \( g(y) \) are convex. The Legendre-Fenchel dual function is defined as

\[ g(y) = \sup_{x \in \mathbb{R}} [yx - f(x)]. \]

\(^1\)In the study book \( \mathcal{P}^{\text{mech}} \) and \( \mathcal{P}^{\text{heat}} \) are denoted as \( \mathcal{P}^{\text{ext}} \) and \( \mathcal{R} \).