Continuum mechanics

2. exercise – mathematical preliminaries

1. (Holzapfel ex. 1 p. 38)
   Let the covariant basis vectors \( g_1, g_2 \) and vector \( u \) be given by
   \[
   g_1 = e_1, \quad g_2 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad u = g_1 + 2g_2,
   \]
   where \( \{e_\alpha\}, \alpha = 1, 2 \) is a set of orthonormal basis vectors.
   Compute the contravariant basis vectors and express \( u \) using the contravariant basis.
   Draw picture and interpret the geometrical situation.

2. (Holzapfel ex. 6 p. 39)
   Let an orthogonal tensor \( Q \) be represented by
   \[
   Q = Q^i_j g_i \otimes g^j = Q^j_i g^i \otimes g_j.
   \]
   (a) Use the definition of the transpose \( v \cdot A^T u = u \cdot A v = A v \cdot u \) to verify that
   \[Q^T = Q^i_j g^j \otimes g_i = Q^j_i g_j \otimes g^i\]
   and show that the orthogonality condition \( QQ^T = I \) may be written as
   \[Q^j_i Q^k_i = \delta^j_k, \quad g_{kl} Q^j_i Q^k_i = g_{ij}.\]
   (b) Show that under the transformation \( \tilde{g}_i = Q g_i, \quad g_i = Q^T \tilde{g}_i, \) \( i = 1, 2, 3 \) the components of the Euclidean metric tensor \( I \) remain unchanged.

3. (Holzapfel ex.2 p. 43)
   Suppose that the tensor \( A(t) \) is invertible. Prove the important relation
   \[
   \dot{\det A(t)} = \det A(t) \text{tr}[A^{-1}(t) \dot{A}(t)] = \det A(t) A^{-T} : \dot{A}(t).
   \]
   The superimposed dot denotes time derivative. Check the result with the matrix representation of the tensor given by
   \[
   [A(t)] = \begin{bmatrix}
   1/2 & 0 & 0 \\
   t^{-\alpha} & 1 & 0 \\
   0 & 0 & 2
   \end{bmatrix},
   \]
   where \( \alpha \) is a scalar.

4. Compute the contravariant components of the metric tensor \( g^{mn} \) in cylindrical coordinate system.