Continuum mechanics

1. exercise – mathematical preliminaries

1. (CMT, ex 2.1 p. 661)

A rocket propels itself forward by burning fuel and emitting the resulting hot gases at high velocity out of a nozzle at the rear of the rocket. As a result of the combustion process the mass of the rocket continuously decreases.

(a) Show that the motion of the rocket is governed by the following equation:

\[ m \frac{dv}{dt} + v_\text{ex} \frac{dm}{dt} = F(t), \]

where \( v = v(t) \) is the velocity of the rocket, \( m = m(t) \) is the mass of the rocket, \( v_\text{ex} \) is the velocity of the exhaust gas relative to that of the rocket, and \( F(t) \) is the external force acting on the rocket. **Hint:** Compute the momentum of the rocket at time \( t \) and time \( t + \Delta t \), i.e. \( p(t) \) and \( p(t + \Delta t) = p + \Delta p \). The mass of the rocket will be reduced by \( \Delta m \) during this interval. Account for the momentum of the exhaust gas. Obtain \( dp/dt \) through a limiting operation.

(b) Compute the maximum velocity, \( v_{\text{max}} \), that the rocket can achieve under the following conditions. There is no external force acting on the rocket, \( F(t) = 0 \), the relative exhaust velocity and rate of change of mass are constant, the initial velocity is zero, the initial mass of the rocket is \( m_{\text{init}} \) the final mass of the rocket after the fuel is expended is \( m_{\text{fin}} \). Given your result, what is the best way for a rocket engineer to increase the maximum velocity?

**Solution:**

(a):

We consider the rocket in two time elapse; \( t \) and \( t + \Delta t \).

\[
\begin{align*}
\Delta m & \quad \Delta m \\
V & \quad V_{\text{ex}} \\
@t & \quad @(t + \Delta t)
\end{align*}
\]

We have:

\[
v_e = (v + \Delta v) - v_\text{ex},
\]

where:

- \( v + \Delta v \): Velocity of rocket in fix frame
- \( v_\text{ex} \): Velocity of exhaust in rocket frame
- \( v_e \): Velocity of exhaust in fix frame

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1CMT: Tadmor, Miller, Elliot: Continuum Mechanics and Thermodynamics
Momentums:

\( P_1 = (m + \Delta m)v, \)

\( P_2 = m(v + \Delta v) + \Delta m \cdot v_e, \)

We have:

\[ \sum F_i = F_{\text{ex}} = \lim_{\Delta t \to 0} \frac{P_2 - P_1}{\Delta t} \]

Thus:

\[ F_{\text{ex}} = m \cdot \frac{dv}{dt} + \frac{dm}{dt} \cdot v_e^* \]

Note: \( \Delta m \to -\frac{dm}{dt} \)

(b):

if \( F(t) = 0 \):

\[ m \cdot \frac{dv}{dt} = -\frac{dm}{dt} \cdot v_e^* \]

Then by integrating:

\[ \int_{v_0}^{v_1} dv = -\int_{m_0}^{m_1} \frac{dm}{m} \cdot v_e^* \]

\[ \implies \Delta v = -v_e^* \cdot \ln \frac{m_1}{m_0} = v_e^* \cdot \ln \frac{m_0}{m_1} \]

if initial speed is zero: \( v_0 = 0 \):

\[ \Delta v = v_1 = v_e^* \cdot \ln \frac{m_0}{m_1} \]

2. (Holzapfel ex.9 p. 20)

Find the axial vector of a skew tensor \( W = \frac{1}{2}(u \otimes v - v \otimes u) \).

**Solution:**

\[ W_{ij} = \frac{1}{2}(u_i v_j - v_i u_j) \]

The axial vector can be calculated by:

\[ w_k = -\frac{1}{2} \cdot \epsilon_{ijk} \cdot W_{ij} \]

Thus:

\[ w_k = -\frac{1}{4} \cdot \epsilon_{ijk} \cdot (u_i v_j - v_i u_j) \]

The skew property: \(-v_i u_j = v_j u_i\)

\[ w_k = -\frac{1}{4} \cdot \epsilon_{ijk} \cdot (u_i v_j + v_j u_i) = \frac{1}{2} \cdot \epsilon_{ijk} \cdot u_i v_j = -\frac{1}{2} u \times v \]

3. (CMT, ex 2.13 p. 68)

Consider the dyad \( D = a \otimes a \) constructed from the vector \( a \).
(a) Write out the components of $D$ in matrix form.

(b) Compute the three principal invariants of $D : I_1, I_2, I_3$. Simplify your expressions as much as possible.

(c) Compute the eigenvalues of $D$.

**Solution:**

(a): Generally: $A = a \otimes b \iff A_{ij} = a_i b_j$

\[
\begin{bmatrix}
  a_1 b_1 & a_1 b_2 & a_1 b_3 \\
  a_2 b_1 & a_2 b_2 & a_2 b_3 \\
  a_3 b_1 & a_3 b_2 & a_3 b_3
\end{bmatrix} \iff
\begin{bmatrix}
  a_1^2 & a_1 a_2 & a_1 a_3 \\
  a_2 a_1 & a_2^2 & a_2 a_3 \\
  a_3 a_1 & a_3 a_2 & a_3^2
\end{bmatrix}
\]

(b):

\[
I_1 = \text{tr}(D) = a_1^2 + a_2^2 + a_3^2
\]

\[
I_2 = \frac{1}{2}[(\text{tr}(D))^2 - \text{tr}(D^2)] = 0
\]

\[
I_3 = \text{det}(D) = 0
\]

(c): Eigenvalues of D: $\text{det}(D - \lambda I) = 0$

\[
\lambda_1 = 0
\]

\[
\lambda_2 = a_1^2 + a_2^2 + a_3^2
\]

4. Assume that $v$ is an arbitrary vector and $\hat{n}$ is a unit vector. Show that

\[
v = (v \cdot \hat{n}) \hat{n} + \hat{n} \times (v \times \hat{n})
\]

**Solution:**

Hints:

\[
a \times b = \epsilon_{kij} a_i b_j \hat{e}_k
\]

\[
a \cdot b = a_i b_j
\]

\[
\epsilon_{ijk} \epsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}
\]

Change the right side of equation into index notation:

\[
n \times (v \times n) = (n_j \hat{e}_i) \times (\epsilon_{jkl} v_l n_k \hat{e}_i) = \epsilon_{jkl} n_i v_j n_k (\hat{e}_i \times \hat{e}_l)
\]

\[
= \epsilon_{jkl} \epsilon_{hm} n_i v_j n_k \hat{e}_h
\]

\[
= (\delta_{jh} \delta_{ki} - \delta_{ji} \delta_{kh}) n_i v_j n_k \hat{e}_h \quad (*)
\]

And:

\[
(v \cdot n) \hat{n} = (v_i n_i) n_h \hat{e}_h \quad (**)\]

Then $(*) + (**)$:

\[
(v_i n_i) n_h + \delta_{jh} \delta_{ki} n_i v_j n_k - \delta_{ji} \delta_{kh} n_i v_j n_k
\]

\[
= (v_i n_i) n_h + n_k v_h n_k - (v_i n_i) n_h = v
\]
5. Derive the vector identity below connecting three arbitrary vectors \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) by the method of vector analysis

\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}
\]

**Solution:**

According to the question 4 hints:

\[
a \times (b \times c) = (a_i \hat{e}_i) \times (\epsilon_{jkl} b_j c_k \hat{e}_l) = \epsilon_{jkl} a_i b_j c_k (\hat{e}_i \times \hat{e}_l)
\]

\[
= \epsilon_{jkl} \epsilon_{hil} a_i b_j c_k \hat{e}_h
\]

\[
= (\delta_{jh} \delta_{ki} - \delta_{ji} \delta_{kh}) a_i b_j c_k \hat{e}_h
\]

By considering the fact that: \( a_i \delta_{ij} = a_j \) and \( a_j \delta_{ij} = a_i \):

\[
= a_i b_j c_k \hat{e}_j - a_i b_k c_i \hat{e}_k
\]

\[
= (a_i c_i)(b_j \hat{e}_j) - (a_i b_i)(c_k \hat{e}_k)
\]

\[
= (a \cdot c)b - (a \cdot b)c
\]