Rakenteiden mekaniikan sovellutuksia

kaavakokoelma

7. toukokuuta 2012

Luku 1

Rakenteiden mekaniikan sovellutuksia, kaavakokoelma, osa 1

1.1 Kerrostalo

Vääntökeskiö:

$$\begin{bmatrix} \sum I_{x_i} & -\sum I_{xy_i} \\ -\sum I_{xy_i} & \sum I_{y_i} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} \sum x_i I_{x_i} - \sum y_i I_{xy_i} \\ -\sum x_i I_{xy_i} + \sum y_i I_{y_i} \end{bmatrix}. \tag{1.1}$$

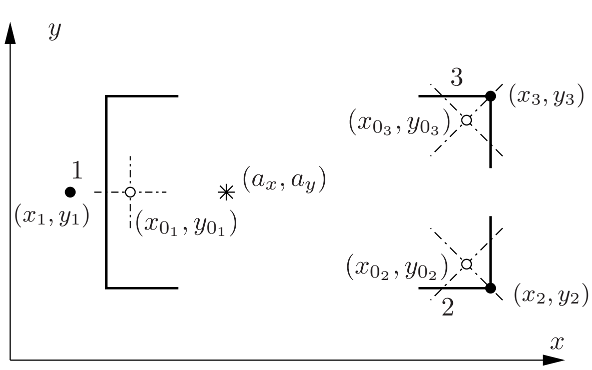
Jos kaikille paarteille pätee

$$I_{xy_i} = 0, \quad i = 1, \ldots, n, \tag{1.2}$$

$$a_x = \frac{\sum_{i=1}^{n} x_i I_{x_i}}{\sum_{i=1}^{n} I_{x_i}}, \quad a_y = \frac{\sum_{i=1}^{n} y_i I_{y_i}}{\sum_{i=1}^{n} I_{y_i}}. \tag{1.3}$$

Momenttien ja muodonmuutossuureiden väliset yhtälöt

$$\begin{bmatrix} M_y \\ M_x \\ B \end{bmatrix} = E \begin{bmatrix} \sum I_{y_i} & \sum I_{xy_i} & 0 \\ -\sum I_{xy_i} & -\sum I_{x_i} & 0 \\ 0 & 0 & -\bar{I}_\omega \end{bmatrix} \begin{bmatrix} v_x'' \\ v_y'' \\ \varphi'' \end{bmatrix}, \tag{1.4}$$



Kuva 1.1 Kerrostalon poikkileikkaus.

$$\bar{I}_\omega = \sum_{i=1}^{n}(x_i - a_x)^2 I_{x_i} + \sum_{i=1}^{n}(y_i - a_y)^2 I_{y_i} - 2\sum_{i=1}^{n}(x_i - a_x)(y_i - a_y) I_{xy_i} + \sum_{i=1}^{n} I_{\omega_i}. \quad (1.5)$$

Muodonmuutossuureet momenttien avulla lausuttuina

$$\begin{bmatrix} v_x'' \\ v_y'' \\ \varphi'' \end{bmatrix} = \frac{1}{ED} \begin{bmatrix} \sum I_{x_i} & \sum I_{xy_i} & 0 \\ -\sum I_{xy_i} & -\sum I_{y_i} & 0 \\ 0 & 0 & -\dfrac{D}{\bar{I}_\omega} \end{bmatrix} \begin{bmatrix} M_y \\ M_x \\ B \end{bmatrix}, \quad (1.6)$$

$$D = (\sum I_{x_i})(\sum I_{y_i}) - (\sum I_{xy_i})^2. \quad (1.7)$$

Paarteen i normaalijännitys

$$\sigma_{z_i} = \frac{(\sum I_{y_i})M_x + (\sum I_{xy_i})M_y}{D}(y - y_{0_i}) - \frac{(\sum I_{x_i})M_y + (\sum I_{xy_i})M_x}{D}(x - x_{0_i}) + \frac{B}{\bar{I}_\omega}\bar{\omega}_i, \quad (1.8)$$

missä $$\bar{\omega}_i(x,y) = (y - y_{0_i})(x_i - a_x) - (x - x_{0_i})(y_i - a_y) + \omega_i(x,y). \quad (1.9)$$

Jos $I_{xy_i} = 0$, $$\sigma_{z_i} = \frac{M_x}{\sum I_{x_i}}(y - y_{0_i}) - \frac{M_y}{\sum I_{y_i}}(x - x_{0_i}) + \frac{B}{\bar{I}_\omega}\bar{\omega}_i. \quad (1.10)$$

Paarteiden taivutusmomentit

$$M_{x_i} = \int_{A_i} \sigma_{z_i}(y - y_{0_i})\, dA, \quad (1.11)$$

$$M_{y_i} = -\int_{A_i} \sigma_{z_i}(x - x_{0_i})\, dA, \quad (1.12)$$

$$M_{x_i} = \frac{(\sum I_{xy_i})I_{x_i} - (\sum I_{x_i})I_{xy_i}}{D} M_y + \frac{(\sum I_{y_i})I_{x_i} - (\sum I_{xy_i})I_{xy_i}}{D} M_x + \frac{I_{x_i}(x_i - a_x) - I_{xy_i}(y_i - a_y)}{\bar{I}_\omega} B, \quad (1.13)$$

$$M_{y_i} = \frac{(\sum I_{x_i})I_{y_i} - (\sum I_{xy_i})I_{xy_i}}{D} M_y + \frac{(\sum I_{xy_i})I_{y_i} - (\sum I_{y_i})I_{xy_i}}{D} M_x + \frac{I_{y_i}(y_i - a_y) - I_{xy_i}(x_i - a_x)}{\bar{I}_\omega} B. \quad (1.14)$$

Jos $I_{xy_i} = 0$, niin

$$M_{x_i} = \frac{I_{x_i}}{\sum I_{x_i}} M_x + \frac{I_{x_i}(x_i - a_x)}{\bar{I}_\omega} B, \quad (1.15)$$

$$M_{yi} = \frac{I_{yi}}{\sum I_{yi}} M_y + \frac{I_{yi}(y_i - a_y)}{\bar{I}_\omega} B, \quad (1.16)$$

$$B_i = \frac{I_{\omega i}}{\bar{I}_\omega} B. \quad (1.17)$$

Leikkausvoimien momentti vääntökeskiön suhteen

$$M_{zs} = \sum_{i=1}^{n} Q_{y_i}(x_i - a_x) - \sum_{i=1}^{n} Q_{x_i}(y_i - a_y) + \sum_{i=1}^{n} B_i', \tag{1.18}$$

$$B_i = -EI_{\omega i}\varphi''. \tag{1.19}$$

Tasapainoehdot

$$Q_{y_i} = M_{x_i}', \quad Q_{x_i} = -M_{y_i}', \quad M_{zs} = B'. \tag{1.20}$$

Kokonaisvääntömomentti

$$M_z = M_{zs} + M_{zv}, \tag{1.21}$$

Momenttien tasapainoyhtälöt

$$M_x'' = -p_y, \quad M_y'' = p_x, \quad M_z' = -m, \tag{1.22}$$

Pääakseliston tasapainoyhtälöt

$$(\sum_{i=1}^{n} EI_{y_i})v_x^{(4)} = p_x, \tag{1.23}$$

$$(\sum_{i=1}^{n} EI_{x_i})v_y^{(4)} = p_y, \tag{1.24}$$

$$E\bar{I}_\omega\varphi^{(4)} - G\bar{I}_v\varphi^{(2)} = m, \tag{1.25}$$

$$\bar{I}_v = \sum_{i=1}^{n} I_{vi}. \tag{1.26}$$

Väännön differentiaaliyhtälö

$$\varphi^{(4)} - \bar{k}^2\varphi^{(2)} = \frac{m}{E\bar{I}_\omega}, \tag{1.27}$$

$$\bar{k} = \sqrt{\frac{G\bar{I}_v}{E\bar{I}_\omega}}. \tag{1.28}$$

Bimomentin avulla

$$B^{(2)} - \bar{k}^2 B = -m. \tag{1.29}$$

Väännön DY:n ratkaisu

$$\varphi(z) = C_1 + C_2 z + C_3 \sinh \bar{k}z + C_4 \cosh \bar{k}z + \varphi_0. \tag{1.30}$$

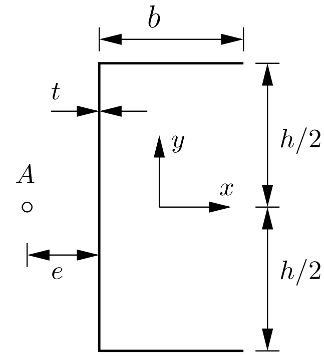
Vääntökulman differentiaaliyhtälön ratkaisu

Differentiaaliyhtälön

$$\frac{d^4\varphi}{dz^4} - k^2\frac{d^2\varphi}{dz^2} = f(z), \quad k^2 = \frac{GI_v}{EI_\omega} \tag{1.31}$$

ratkaisu on

$$\varphi(z) = C_1 + C_2 z + C_3 \sinh kz + C_4 \cosh kz + \varphi_0 \tag{1.32}$$



Kuva 1.2 U-poikkileikkaus.

eli

$$\varphi(z) = \varphi_h(z) + \varphi_0(z). \tag{1.33}$$

Resultantit

$$M_v = GI_v \frac{d\varphi}{dz} = GI_v \left(C_2 + C_3 k \cosh kz + C_4 k \sinh kz + \frac{d\varphi_0}{dz} \right), \tag{1.34}$$

$$B = -EI_\omega \frac{d^2\varphi}{dz^2} = -GI_v \left(C_3 \sinh kz + C_4 \cosh kz + \frac{1}{k^2} \frac{d^2\varphi_0}{dz^2} \right) \tag{1.35}$$

$$M_\omega = \frac{dB}{dz} = -GI_v \left(C_3 k \cosh kz + C_4 k \sinh kz + \frac{1}{k^2} \frac{d^3\varphi_0}{dz^3} \right), \tag{1.36}$$

$$M_z = M_v + M_\omega = GI_v \left(C_2 + \frac{d\varphi_0}{dz} - \frac{1}{k^2} \frac{d^3\varphi_0}{dz^3} \right). \tag{1.37}$$

U-poikkileikkaus

Kuvan 1.2 poikkileikkaukselle

$$e = \frac{b}{2 + \dfrac{h}{3b}}, \quad I_\omega = e^2 I_x + \tfrac{1}{6}(b - 3e)h^2 b^2 t, \quad I_x = \int_A y^2 dA. \tag{1.38}$$

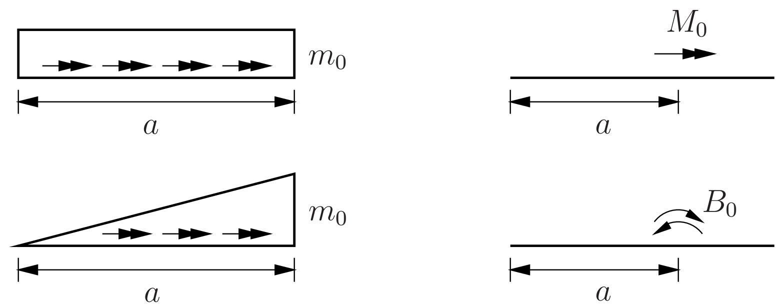
Eräitä yksityisratkaisuja

1. Tasaisen kuorman (kuva 1.3a), $m(z) = m_0$, yksityisratkaisu on

$$\varphi_0 = -\frac{m_0}{2} \frac{1}{GI_v} z^2. \tag{1.39}$$

2. Lineaarisesti jakautuneelle kuormalle (kuva 1.3b) $m(z) = m_0 \dfrac{z}{L}$

$$\varphi_0 = -\frac{m_0}{6L} \frac{1}{GI_v} z^3. \tag{1.40}$$



Kuva 1.3 Kuormitustapauksia.

3. Pistemäinen vääntömomentti M_0 kohdassa $z = a$ (kuva 1.3c):

$$\varphi_0 = 0, \text{ kun } z < a, \tag{1.41}$$

$$\varphi_0 = \frac{M_0}{kGI_v}\left[\sinh k(z-a) - k(z-a)\right], \text{ kun } z > a. \tag{1.42}$$

4. Pistemäinen bimomentti kohdassa $z = a$ (kuva 1.3d):

$$\varphi_0 = 0, \text{ kun } z < a, \tag{1.43}$$

$$\varphi_0 = \frac{B_0}{GI_v}\left[\cosh k(z-a) - 1\right], \text{ kun } z > a. \tag{1.44}$$

1.2 Jäykistetty sauva

Siteen leikkausvoiman $X_k = 1$ siirtymäero δ_{ik} kohdassa i

Poikkisiteen taipumasta siteeseen k

$$\delta^a_{kk} = 2\left[\frac{1}{3EI_s}\left(\frac{a}{2}\right)^3 + \frac{\zeta}{GA_s}\left(\frac{a}{2}\right)\right] = \frac{a^3}{12EI_s} + \frac{a\zeta}{GA_s}, \tag{1.45}$$

siirtymäkerroin suorakaiteelle $\zeta = 1.2$. Poikkisiteessä vaikuttavan kuorman $X_k = 1$ bimomentti vaikutuskohdassa:

$$B_k = \sum P_i\omega_i = -1\omega_1 + 1\omega_2 = \omega_2 - \omega_1 = 2\Omega. \tag{1.46}$$

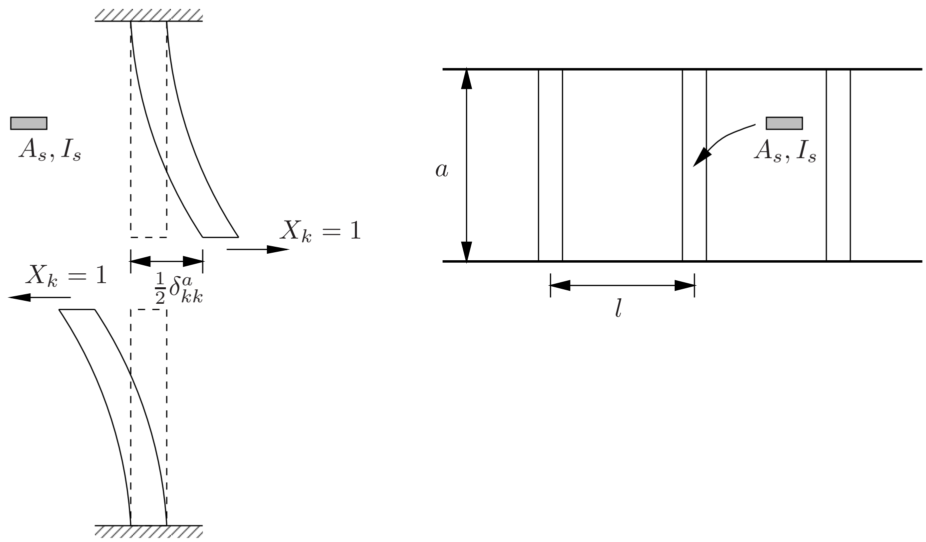
Pistebimomentin B_k staattisesti määrätyn perusmuodon siirtymäero

$$\delta^b_{ik} = w_2 - w_1 = -(\omega_2 - \omega_1)\varphi'_B(z_i) = -2\Omega\varphi'_B(z_i). \tag{1.47}$$

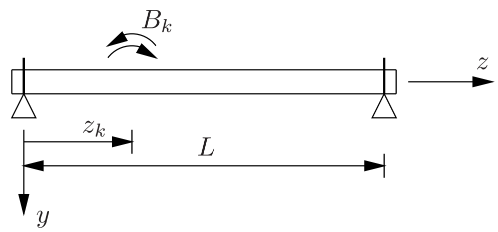
Vapaasti tuetun sauvan tapauksessa

$$\delta^b_{ik} = \frac{(2\Omega)^2}{LGI_v}\left\{kL\frac{\cosh(kz_i)\cosh[k(L-z_k)]}{\sinh kL} - 1\right\}, \quad \text{kun} \quad i \le k, \tag{1.48}$$

$$\delta^b_{ik} = \frac{(2\Omega)^2}{LGI_v}\left\{kL\frac{\cosh[k(L-z_i)]\cosh(kz_k)}{\sinh kL} - 1\right\}, \quad \text{kun} \quad i \ge k, \tag{1.49}$$



Kuva 1.4 Poikkisiteen taipuma kuormasta $X_k = 1$.



Kuva 1.5 Bimomentti kohdassa z_k.

$$k = \sqrt{\frac{GI_v}{EI_\omega}}. \tag{1.50}$$

Siirtymäero siteen aukileikkauskohdassa

$$\delta_{ik} = \delta_{ik}^a + \delta_{ik}^b, \tag{1.51}$$

$$\delta_{ik}^a = 0, \quad \text{jos} \quad i \neq k. \tag{1.52}$$

Yhteensopivuusehdot siteiden kohdalla muodossa

$$\sum_{j=1}^{N} \delta_{ij} X_j + \delta_{i0} = 0, \quad \text{kun} \quad i = 1, \ldots, N. \tag{1.53}$$

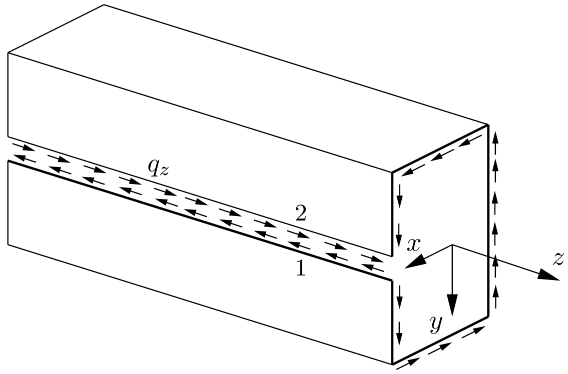
1.2.1 Ekvivalentti levyjäykiste

Keskimääräinen jakautunut bimomentti

$$b = \frac{B_i X_i}{l} = 2\Omega \frac{X_i}{l}, \tag{1.54}$$

missä l on siteiden välimatka.
Väännön differentiaaliyhtälö

$$EI_\omega \varphi^{(4)} - GI_v \varphi'' = m_z + b' = m_z + 2\Omega q'(z). \tag{1.55}$$



Kuva 1.6 Ekvivalentin levyn leikkausvuo.

Leikkausvuota $q(z)$ vastaava vääntömomentti

$$M_z^q = 2\Omega q(z). \tag{1.56}$$

Yhteensopivuusehto

$$\delta q(z) = 2\Omega\varphi', \tag{1.57}$$

$$\delta = \oint \frac{ds}{Gt(s)}. \tag{1.58}$$

Leikkausvuo

$$q(z) = \frac{2\Omega}{\delta}\varphi' = \frac{GI_v^0}{2\Omega}\varphi', \tag{1.59}$$

$$I_v^0 = \frac{4\Omega^2}{G\delta}. \tag{1.60}$$

Väännön differentiaaliyhtälö

$$EI_\omega\varphi^{(4)} - G(\bar{I}_v)\varphi'' = m_z(z), \tag{1.61}$$

$$\bar{I}_v = I_v + I_v^0. \tag{1.62}$$

Vääntömomentti M_v (Saint Venantin vääntö)

$$M_v(z) = G(I_v + I_v^0)\varphi''(z). \tag{1.63}$$

1.2.2 Energiamenetelmä

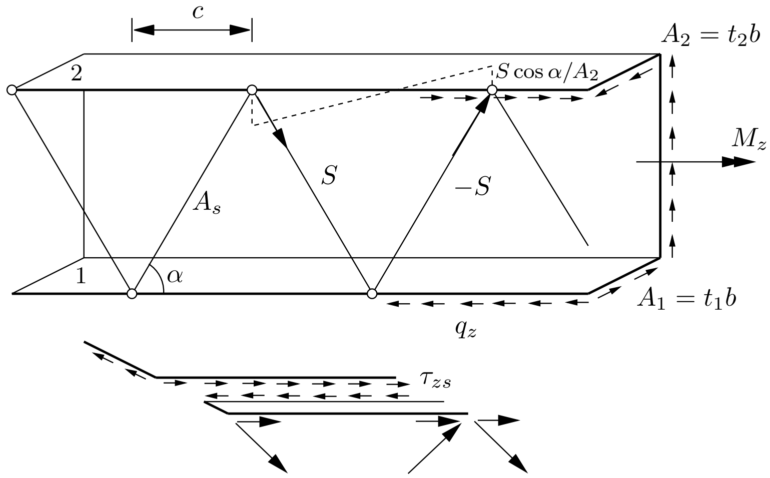
Komplementaarisen energian minimin periaate

Vääntöjäykkyys

$$GI_v = \frac{M_z(\sigma)}{\theta} = \frac{M_z^2}{2W(\sigma)}, \tag{1.64}$$

$W = W(\sigma)$ on luvallisesta jännitystilasta riippuva jännitysenergia. Ortogonaaliset jännitystilat

$$W(\sigma_i, \sigma_k) = \begin{cases} 0, & i \neq k, \\ W(\sigma_i), & i = k, \end{cases} \tag{1.65}$$



Kuva 1.7 Ristikolla jäykistetty U-profiili.

vääntöjäykkyyden alaraja-arvio

$$GI_{v\sigma} = \sum_{i=1}^{n} \frac{M_z(\sigma_i)}{\theta} = \sum_{i=1}^{n} \frac{[M_z(\sigma_i)]^2}{2W(\sigma_i)} = \sum_{i=1}^{n} GI_{v\sigma_i}. \tag{1.66}$$

Ristikolla jäykistetyn sauvan vääntömomentti Bredt'in kaavan mukaan

$$M_z = 2\Omega q(z) \tag{1.67}$$

Sauvan komplementaarinen energia pituusyksikköä kohti

$$\bar{W} = \frac{1}{2}\left[\frac{S^2 a}{lEA_s\sin\alpha} + \frac{A_1}{l}\int_0^l \frac{\sigma_z^2}{E}\,dz + \frac{A_2}{l}\int_0^l \frac{\sigma_z^2}{E}\,dz + \oint \frac{\tau^2}{G}t(s)\,ds\right]. \tag{1.68}$$

Ortogonaaliset jännitystilat:

- tila 1 : U-profiilin vääntöjännitykset,
- tila 2 : poikkisiteiden sauvavoimista aiheutuvat jännitykset.

Potentiaalienergian minimin periaatte

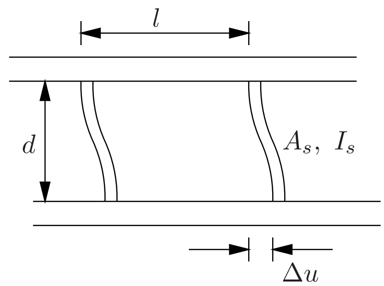
$$GI_{v\varepsilon} = \frac{\bar{M}_z}{\theta} = \frac{2W(\varepsilon)}{\theta^2}, \tag{1.69}$$

missä M_z on annettu vääntömomentti.
Kinemaattisesti luvallisten siirtymätilojen yhdistelmä

$$u = u_1 + u_2 + \cdots + u_n, \tag{1.70}$$

jota vastaa muodonmuutostila ε ja vääntymä θ. Ortogonaaliset muodonmuutostilat ε_i

$$W(\varepsilon_i, \varepsilon_k) = \begin{cases} 0, & i \neq k, \\ W(\varepsilon_i), & i = k, \end{cases} \tag{1.71}$$



Kuva 1.8 Poikkisitein liitetyt sauvat.

vääntöjäykkyyden yläraja-arvio

$$GI_{v\varepsilon} = \frac{\bar{M}_z}{\sum\limits_i \theta_i} = \frac{1}{\sum\limits_i \dfrac{\theta_i^2}{2W(\varepsilon_i)}} = \frac{1}{\sum\limits_i \dfrac{1}{GI_v(\varepsilon_i)}}. \tag{1.72}$$

1.3 Kerrospalkki

Liitoksen liukuman ja leikkausvuon $q(x)$ yhteys

$$q(x) = K\Delta u(x), \tag{1.73}$$

$$K = \frac{12EI_s}{ld^3\left[1 + \zeta_s \dfrac{E}{G}\left(\dfrac{h}{d}\right)^2\right]}, \tag{1.74}$$

missä $\zeta_s = 1.2$ on suorakaidepoikkileikkaukselle, E ja G ovat poikkisiteen kimmokerroin ja liukumoduuli ja h on poikkisiteen korkeus. Paarteen normaalivoiman differentiaaliyhtälö

$$\frac{d^4N_1(x)}{dx^4} - \alpha^2\frac{d^2N_1(x)}{dx^2} = -\beta p(x), \tag{1.75}$$

$$\alpha^2 = K\left(\frac{c^2}{EI_0} + \frac{EA_0}{EA_p}\right), \tag{1.76}$$

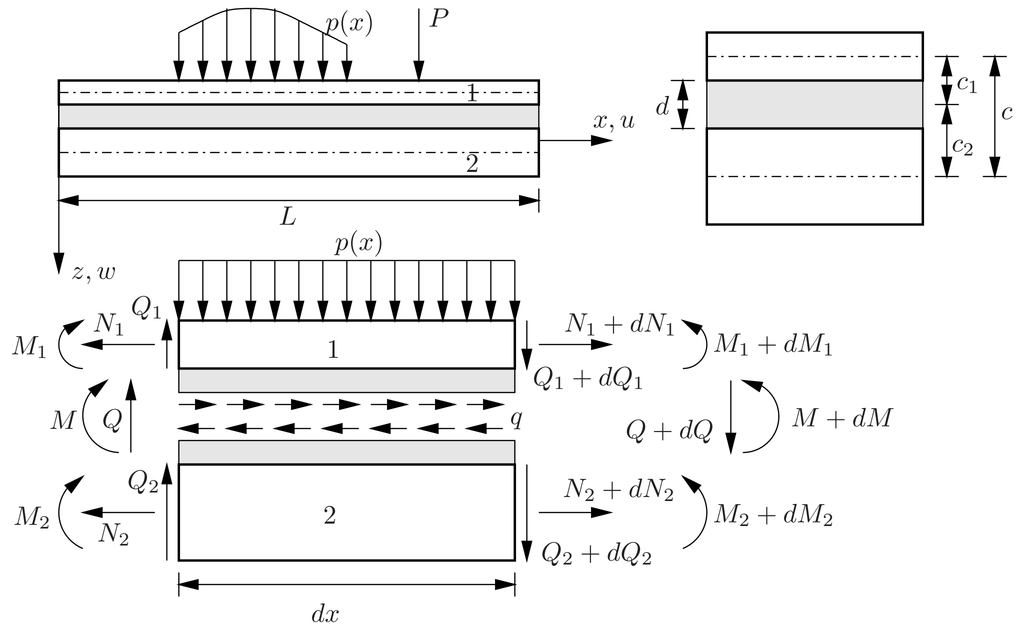
$$EA_0 = EA_1 + EA_2, \quad EA_p = EA_1 \cdot EA_2, \tag{1.77}$$

$$\beta = \frac{Kc}{EI_0}, \quad EI_0 = EI_1 + EI_2. \tag{1.78}$$

Paarteen normaalivoiman differentiaaliyhtälö taipuman $w(x)$ avulla

$$\frac{d^4w(x)}{dx^4} - \alpha^2\frac{d^2w(x)}{dx^2} = \alpha^2\frac{M(x)}{EI} - \frac{1}{EI_0}\frac{d^2M(x)}{dx^2}, \tag{1.79}$$

$$EI = EI_0 + \frac{EA_p}{EA_0}c^2. \tag{1.80}$$



Kuva 1.9 Kerrospalkin voimasuureet.

1.3.1 Tasapainoyhtälön ratkaisu

Differentiaaliyhtälön

$$\frac{d^6 w(x)}{dx^6} - \alpha^4 \frac{d^4 w(x)}{dx^2} = -\alpha^2 \frac{p(x)}{EI} + \frac{1}{EI_0}\frac{d^2 p(x)}{dx^2} \tag{1.81}$$

ratkaisu

$$w(x) = w_h(x) + w_p(x), \tag{1.82}$$

$$w_h(x) = C_1 \sinh \alpha x + C_2 \cosh \alpha x + C_3 x^3 + C_4 x^2 + C_5 x + C_6, \tag{1.83}$$

$$w_p(x) = \frac{1}{\alpha^5 EI}\int_0^x \left[\alpha^2 p(s) - \frac{EI}{EI_0}\frac{d^2 p(s)}{ds^2}\right]\left\{\alpha(x-s) + \frac{1}{6}\alpha^3 (x-s)^3 - \sinh[\alpha(x-s)]\right\} ds. \tag{1.84}$$

1.3.2 Reunaehdot

1. Nivelreuna

$$w(0) = M_i(0) = N_1(0) = w(L) = M_i(L) = N_1(L) = 0, \quad i = 1, 2 \tag{1.85}$$

eli

$$w(0) = \frac{d^2 w(0)}{dx^2} = \frac{d^4 w(0)}{dx^4} - \frac{p(0)}{EI_0} = 0 \tag{1.86}$$

$$w(L) = \frac{d^2 w(L)}{dx^2} = \frac{d^4 w(L)}{dx^4} - \frac{p(L)}{EI_0} = 0. \tag{1.87}$$

Edellä $N_1 = -N_2$.

2. Jäykkä tuki pisteessä $x = a$:

$$w(a) = \frac{dw(a)}{dx} = \Delta u(a) = 0, \tag{1.88}$$

joista kolmas voidaan muuntaa muotoon

$$\frac{d^5w(a)}{dx^5} - \alpha^2 \left(1 - \frac{EI_0}{EI}\right) \frac{d^3w(a)}{dx^3} = \frac{1}{EI_0}\frac{dp(a)}{dx}. \tag{1.89}$$

3. Vapaa reuna on kohdassa $x = a$:

$$M_i(a) = N_1(a) = Q(a) = 0, \quad i = 1, 2. \tag{1.90}$$

Taipuman avulla

$$\frac{d^2w(a)}{dx^2} = 0, \tag{1.91}$$

$$\frac{d^4w(a)}{dx^4} = \frac{p(a)}{EI_0}, \tag{1.92}$$

$$\frac{d^5w(a)}{dx^5} - \alpha^2 \frac{d^3w(a)}{dx^3} = \frac{1}{EI_0}\frac{dp(a)}{dx}. \tag{1.93}$$

1.3.3 Voimasuureet

$$\begin{aligned} M(x) =& M_1(x) + M_2(x) - cN_1(x) \\ =& \frac{EI}{\alpha^2}\frac{d^4w(x)}{dx^4} - EI\frac{d^2w(x)}{dx^2} - \frac{EI}{\alpha^2 EI_0}p(x), \end{aligned} \tag{1.94}$$

$$Q(x) = Q_1(x) + Q_2(x) = \frac{dM(x)}{dx}, \tag{1.95}$$

$$N_1(x) = -N_2(x) = -\frac{1}{c}\left[M(x) + EI_0\frac{d^2w(x)}{dx^2}\right], \tag{1.96}$$

$$q(x) = k\Delta u(x) = -\frac{dN_1(x)}{dx} = \frac{dN_2(x)}{dx}, \tag{1.97}$$

$$M_i(x) = -EI_i\frac{d^2w(x)}{dx^2}, \quad i = 1, 2, \tag{1.98}$$

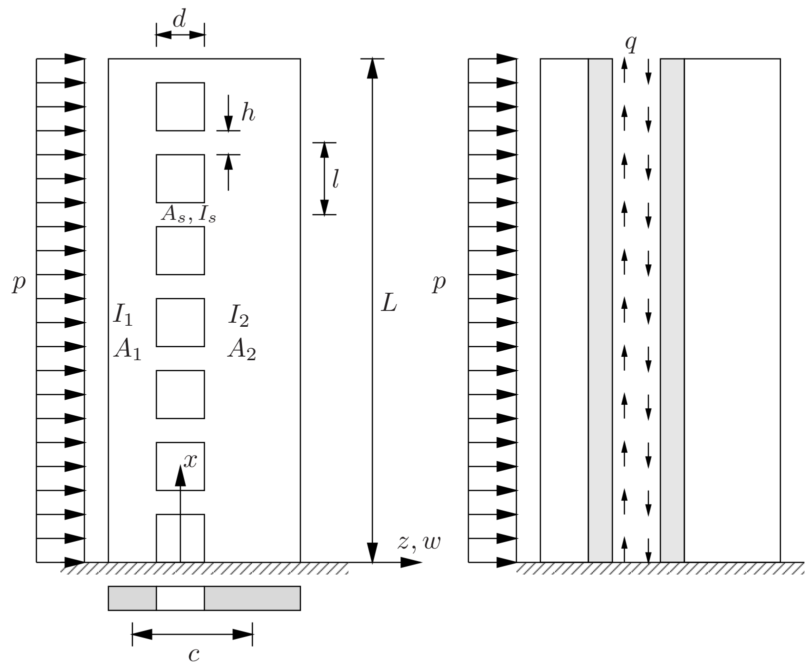
$$Q_i(x) = \frac{dM_i(x)}{dx} + c_i q(x), \quad i = 1, 2. \tag{1.99}$$

1.3.4 Aukollinen tukiseinä

Paarteen normaalivoiman DY

$$\frac{d^2N_1(x)}{dx^2} - \alpha^2 N_1(x) = \beta M(x). \tag{1.100}$$

Paarteen normaalivoima $T = N_1$



Kuva 1.10 Aukollinen tukiseinä.

$$\frac{dT(x)}{dx} = -q(x), \tag{1.101}$$

paarteen normaalivoiman DY myös

$$\frac{d^2T(x)}{dx^2} - \alpha^2 T(x) = \beta M(x), \tag{1.102}$$

$$\alpha^2 = K\left(\frac{c^2}{EI_0} + \frac{EA_0}{EA_p}\right), \tag{1.103}$$

$$\beta = \frac{Kc}{EI_0}. \tag{1.104}$$

Tasaisen kuorman, $p =$ vakio, tapauksessa

$$M(x) = -\frac{1}{2}p(L-x)^2, \tag{1.105}$$

$$T(x) = C_1 \sinh \alpha x + C_2 \cosh \alpha x + T_p(x), \tag{1.106}$$

$$T_p(x) = -\frac{1}{\alpha^2}\left[1 + \frac{D^2}{\alpha^2} + \frac{D^4}{\alpha^4} + \cdots\right]\beta M(x), \tag{1.107}$$

$$D \equiv \frac{d}{dx}. \tag{1.108}$$

Ulokkeen reunaehdot

$$\frac{dw(0)}{dx} = 0, \quad q(0) = 0, \quad \frac{dT(0)}{dx} = 0, \quad T(L) = 0. \tag{1.109}$$

Paarteiden momentit

$$M_i(x) = \frac{I_i}{I_0}\left[M(x) + cT(x)\right], \quad i = 1, 2. \tag{1.110}$$

Taipuman differentiaaliyhtälö

$$\frac{d^4w(x)}{dx^4} - \alpha^2\frac{d^2w(x)}{dx^2} = \alpha^2\frac{M(x)}{EI} - \frac{1}{EI_0}\frac{d^2M(x)}{dx^2}, \tag{1.111}$$

$$EI_0 = EI_1 + EI_2, \tag{1.112}$$

$$\begin{aligned}\alpha^2 =& K\left(\frac{1}{EA_1} + \frac{1}{EA_2} + \frac{c^2}{EI_0}\right)\\ =& K\left(\frac{EA_0}{EA_p} + \frac{c^2}{EI_0}\right),\end{aligned} \tag{1.113}$$

$$\beta = \frac{Kc}{EI_0}, \tag{1.114}$$

$$EA_0 = EA_1 + EA_2, \quad EA_p = EA_1 \cdot EA_2, \tag{1.115}$$

$$EI = EI_0 + \frac{EA_p}{EA_0}c^2. \tag{1.116}$$

Taipuman neljännen kertaluvun differentiaaliyhtälön ratkaisu

$$w(x) = C_1 + C_2x + C_3\cosh\alpha x + C_4\sinh\alpha x + w_p(x), \tag{1.117}$$

yksityisratkaisu

$$w_p(x) = \frac{1}{\alpha^2 EI_0}\left[\frac{1}{D^2} + \frac{1}{\alpha^2} + \frac{D^2}{\alpha^4} + \frac{D^4}{\alpha^6} + \cdots\right]\left[\frac{d^2M(x)}{dx^2} - \alpha^2\frac{EI_0}{EI}M(x)\right]. \tag{1.118}$$

Ulokepalkin reunaehdot

$$w(0) = 0, \quad \frac{dw(0)}{dx} = 0, \quad \frac{d^2w(L)}{dx^2} = 0, \tag{1.119}$$

$$\frac{d^3w(L)}{dx^3} - \alpha^2\frac{dw(L)}{dx} = \frac{\alpha^2}{EI}\int_0^L M(\xi)\,d\xi - \frac{1}{EI_0}\frac{dM(L)}{dx}. \tag{1.120}$$

Luku 2

Rakenteiden mekaniikan sovellutuksia, kaavakokoelma, osa 2

2.1 Ympyrälaatta

Sylinterikoordinaatit (r, φ, z) ja suorakulmaiset koordinaatit (x, y, z):

$$x = r\cos\varphi, \quad y = r\sin\varphi, \tag{2.1}$$

$$r^2 = x^2 + y^2, \quad \varphi = \arctan\left(\frac{y}{x}\right), \tag{2.2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\varphi, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\varphi, \tag{2.3}$$

$$\frac{\partial \varphi}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\varphi}{r}, \quad \frac{\partial \varphi}{\partial y} = \frac{x}{r^2} = \frac{\cos\varphi}{r}. \tag{2.4}$$

Laplacen operaattori

$$\nabla^2(\bullet) = \frac{\partial^2(\bullet)}{\partial x^2} + \frac{\partial^2(\bullet)}{\partial y^2}, \tag{2.5}$$

$$\nabla^2 w(x,y) = \frac{\partial^2 w(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\varphi)}{\partial \varphi^2}. \tag{2.6}$$

Laatan taipuman differentiaaliyhtälö

$$\nabla^2[\nabla^2 w(x,y)] = \frac{p(x,y)}{D} \tag{2.7}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left[\frac{\partial^2 w(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\varphi)}{\partial \varphi^2}\right] = \frac{p(r,\varphi)}{D}, \tag{2.8}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{2.9}$$

Käyristymät

$$\kappa_r = -\frac{\partial^2 w(r,\varphi)}{\partial r^2}, \tag{2.10}$$

$$\kappa_\varphi = -\frac{1}{r}\frac{\partial w(r,\varphi)}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w(r,\varphi)}{\partial \varphi^2}, \tag{2.11}$$

$$\kappa_{r\varphi} = \frac{1}{r^2}\frac{\partial w(r,\varphi)}{\partial \varphi} - \frac{1}{r}\frac{\partial^2 w(r,\varphi)}{\partial r \partial \varphi} = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial w(r,\varphi)}{\partial \varphi}\right]. \tag{2.12}$$

Muodonmuutokset

$$\varepsilon_r(r,\varphi,z) = z\kappa_r(r,\varphi), \quad \varepsilon_\varphi(r,\varphi,z) = z\kappa_\varphi(r,\varphi), \quad \gamma_{r\varphi}(r,\varphi,z) = 2z\kappa_{r\varphi}(r,\varphi). \tag{2.13}$$

Taivutusmomentit

$$M_r = \int_{-h/2}^{h/2} z\sigma_r\,dz, \quad M_\varphi = \int_{-h/2}^{h/2} z\sigma_\varphi\,dz, \quad M_{r\varphi} = \int_{-h/2}^{h/2} z\tau_{r\varphi}\,dz. \tag{2.14}$$

Tasojännitystilassa

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\varphi), \tag{2.15}$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}(\varepsilon_\varphi + \nu\varepsilon_r), \tag{2.16}$$

$$\tau_{r\varphi} = G\gamma_{r\varphi}, \quad G = \frac{E}{2(1+\nu)}. \tag{2.17}$$

Momenttien ja käyristymien väliset kaavat

$$M_r = D(\kappa_r + \nu\kappa_\varphi), \tag{2.18}$$

$$M_\varphi = D(\kappa_\varphi + \nu\kappa_r), \tag{2.19}$$

$$M_{r\varphi} = D(1-\nu)\kappa_{r\varphi}, \tag{2.20}$$

$$M_r = -D\left\{\frac{\partial^2 w(r,\varphi)}{\partial r^2} + \nu\left[\frac{1}{r}\frac{\partial w(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\varphi)}{\partial \varphi^2}\right]\right\}, \tag{2.21}$$

$$M_\varphi = -D\left[\frac{1}{r}\frac{\partial w(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\varphi)}{\partial \varphi^2} + \nu\frac{\partial^2 w(r,\varphi)}{\partial r^2}\right], \tag{2.22}$$

$$\begin{aligned} M_{r\varphi} &= -D(1-\nu)\left[\frac{1}{r^2}\frac{\partial w(r,\varphi)}{\partial \varphi} - \frac{1}{r}\frac{\partial^2 w(r,\varphi)}{\partial r \partial \varphi}\right] \\ &= -D(1-\nu)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial w(r,\varphi)}{\partial \varphi}\right]. \end{aligned} \tag{2.23}$$

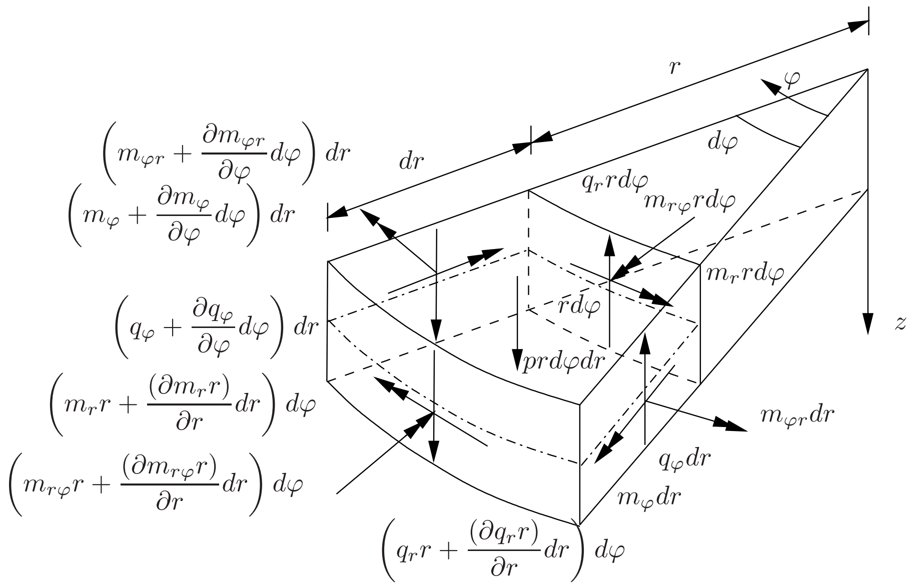
Tasapainoyhtälöt

$$\frac{\partial(Q_r r)}{\partial r} + \frac{\partial Q_\varphi}{\partial \varphi} + p(r,\varphi)r = 0, \tag{2.24}$$

$$\frac{\partial(M_r r)}{\partial r} + \frac{\partial M_{\varphi r}}{\partial \varphi} - M_\varphi - Q_r r = 0, \tag{2.25}$$

$$\frac{\partial M_\varphi}{\partial \varphi} + \frac{\partial(M_{r\varphi} r)}{\partial r} + M_{\varphi r} - Q_\varphi r = 0, \tag{2.26}$$

$$\Rightarrow \frac{\partial^2(M_r r)}{\partial r^2} + \frac{2}{r}\frac{\partial^2(rM_{r\varphi})}{\partial r \partial \varphi} - \frac{\partial M_\varphi}{\partial r} + \frac{1}{r}\frac{\partial^2 M_\varphi}{\partial \varphi^2} + p(r,\varphi)r = 0. \tag{2.27}$$



Kuva 2.1 Laatan alkion tasapaino.

$$Q_r = \frac{\partial M_r}{\partial r} + \frac{1}{r} M_r + \frac{1}{r}\frac{\partial M_{\varphi r}}{\partial \varphi} - \frac{1}{r} M_{\varphi}, \tag{2.28}$$

$$Q_{\varphi} = \frac{1}{r}\frac{\partial M_{\varphi}}{\partial \varphi} + \frac{\partial M_{r\varphi}}{\partial r} + \frac{2}{r} M_{r\varphi}, \tag{2.29}$$

$$Q_r = -D\frac{\partial}{\partial r}\left[\frac{\partial^2 w(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\varphi)}{\partial \varphi^2}\right] = -D\frac{\partial \Delta w(r,\varphi)}{\partial r}, \tag{2.30}$$

$$Q_{\varphi} = -D\frac{1}{r}\frac{\partial}{\partial \varphi}\left[\frac{\partial^2 w(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,\varphi)}{\partial \varphi^2}\right] = -D\frac{1}{r}\frac{\partial \Delta w(r,\varphi)}{\partial \varphi}, \tag{2.31}$$

$$\text{Laplace} \quad \Delta(\bullet) = \nabla^2(\bullet). \tag{2.32}$$

Korvikeleikkausvoima reunalla $r =$ vakio ja reunalla $\varphi =$ vakio

$$\begin{aligned} V_r =& Q_r + \frac{1}{r}\frac{\partial M_{r\varphi}}{\partial \varphi} \\ =& -D\left\{\frac{\partial}{\partial r}[\nabla^2 w(r,\varphi)] + \frac{1-\nu}{r}\frac{\partial}{\partial \varphi}\left[\frac{1}{r}\frac{\partial^2 w(r,\varphi)}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial w(r,\varphi)}{\partial \varphi}\right]\right\}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} V_{\varphi} =& Q_{\varphi} + \frac{\partial M_{r\varphi}}{\partial r} \\ =& -D\left\{\frac{1}{r}\frac{\partial}{\partial \varphi}[\nabla^2 w(r,\varphi)] + (1-\nu)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2 w(r,\varphi)}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial w(r,\varphi)}{\partial \varphi}\right]\right\}. \end{aligned} \tag{2.34}$$

Kimmoisen laatan jännityskomponentit

$$\sigma_r = \frac{12 M_r}{h^3} z, \quad \sigma_{\varphi} = \frac{12 M_{\varphi}}{h^3} z, \quad \tau_{r\varphi} = \frac{12 M_{r\varphi}}{h^3} z. \tag{2.35}$$

2.1.1 Pyörähdyssymmetrinen taivutus

$$\nabla^4 w = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left[\frac{d^2w(r)}{dr^2} + \frac{1}{r}\frac{dw(r)}{dr}\right] = \frac{p(r)}{D}. \tag{2.36}$$

$$M_r(r) = -D\left[\frac{d^2w(r)}{dr^2} + \frac{\nu}{r}\frac{dw(r)}{dr}\right], \tag{2.37}$$

$$M_\varphi(r) = -D\left[\frac{1}{r}\frac{dw(r)}{dr} + \nu\frac{d^2w(r)}{dr^2}\right], \tag{2.38}$$

$$\begin{aligned} Q_r(r) =& -D\frac{d}{dr}\left[\frac{d^2w(r)}{dr^2} + \frac{1}{r}\frac{dw(r)}{dr}\right] \\ =& -D\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{dw(r)}{dr}\right]\right\}, \end{aligned} \tag{2.39}$$

$$\sigma_r(r,z) = -\frac{E}{1-\nu^2}\,z\left[\frac{d^2w(r)}{dr^2} + \frac{\nu}{r}\frac{dw(r)}{dr}\right], \tag{2.40}$$

$$\sigma_\varphi(r,z) = -\frac{E}{1-\nu^2}\,z\left[\frac{1}{r}\frac{dw(r)}{dr} + \nu\frac{d^2w(r)}{dr^2}\right]. \tag{2.41}$$

Identiteetti

$$\begin{aligned} \nabla^2 w(r) =& \frac{d^2w(r)}{dr^2} + \frac{1}{r}\frac{dw(r)}{dr} \\ =& \frac{1}{r}\frac{d}{dr}\left[r\frac{dw(r)}{dr}\right], \end{aligned} \tag{2.42}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{dw(r)}{dr}\right]\right\}\right) = \frac{p(r)}{D}, \tag{2.43}$$

$$w(r) = \int_0^r \frac{1}{r}\int_0^r r\int_0^r \frac{1}{r}\int_0^r \frac{rp(r)}{D}\,dr\,dr\,dr\,dr. \tag{2.44}$$

Tasaisen kuorman, $p(r) = p_0$, tapauksessa

$$\begin{aligned} w(r) =& w_h(r) + w_p(r) \\ =& C_1\ln r + C_2r^2\ln r + C_3r^2 + C_4 + \frac{p_0r^4}{64D}. \end{aligned} \tag{2.45}$$

2.2 Energiamenetelmä

2.2.1 Virtuaalisen työn periaate

Ohuen laatan teoriassa

$$u = -z\frac{\partial w}{\partial x}, \quad v = -z\frac{\partial w}{\partial y}, \quad w = w(x,y), \tag{2.46}$$

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2}, \quad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}, \tag{2.47}$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0. \tag{2.48}$$

Virtuaaliset muodonmuutokset

$$\delta\varepsilon_x = z\delta\kappa_x, \quad \delta\varepsilon_y = z\delta\kappa_y, \quad \delta\gamma_{xy} = 2z\delta\kappa_{xy}. \tag{2.49}$$

Sisäisen virtuaalisen työn lauseke

$$\begin{aligned} \delta W^s =& \int_V [\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy}]\, dV \\ =& \int_A [M_x\delta\kappa_x + M_y\delta\kappa_y + 2M_{xy}\delta\kappa_{xy}]\, dA, \end{aligned} \tag{2.50}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} z \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz, \tag{2.51}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}. \tag{2.52}$$

Laatan ulkoinen virtuaalinen työ

$$\delta W^u = -\int_A p(x,y)\delta w(x,y)\, dA - \int_{S_1}\int_{-h/2}^{h/2} (t_x\delta u + t_y\delta v + t_z\delta w)\, dz\, ds, \tag{2.53}$$

missä $p(x,y)$ on kohtisuoraan laatan tasoa vastaan vaikuttavan jakautuneen kuorman intensiteetti, t_x, t_y ja t_z ovat laatan reunan osalla $S_1 \times h$ annetut (jakautuneet) kuormat. Reunanviivan osalla $S_2 = S - S_1$ tunnetaan siirtymät.

2.2.2 Potentiaalienergian minimin periaate

Muodonmuutosenergia

$$\begin{aligned} U =& \frac{1}{2}\int_A \int_{-h/2}^{h/2} (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})\, dA\, dz \\ =& \frac{1}{2}\int_A (M_x\kappa_x + M_y\kappa_y + 2M_{xy}\kappa_{xy}) dA, \end{aligned} \tag{2.54}$$

$$\begin{aligned} M_x =& -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \\ M_y =& -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \\ M_{xy} =& -D(1-\nu)\frac{\partial^2 w}{\partial x \partial x}, \end{aligned} \tag{2.55}$$

$$\begin{aligned} U =& \frac{1}{2} D \int_A \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\nu) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dA \\ =& \frac{1}{2} D \int_A \left\{ (\Delta w)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dA. \end{aligned} \tag{2.56}$$

Muodonmuutosenergian lausekkeen osa

$$\begin{aligned} I =& 2 \int_A \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx\, dy \\ =& \int_S \left[\frac{\partial w}{\partial n} \left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{\rho} \frac{\partial w}{\partial n} \right) + \frac{\partial w}{\partial s} \left(-\frac{\partial^2 w}{\partial n \partial s} + \frac{1}{\rho} \frac{\partial w}{\partial s} \right) \right] ds, \end{aligned} \tag{2.57}$$

missä ρ on reunaviivan kaarevuussäde.

Jäykästi kiinnitetyllä laatalla $I = 0$. Vapaasti tuetulla laatalla, jonka reuna koostuu suorista osista, $I = 0$. Näissä tapauksissa muodonmuutosenergia on

$$U = \frac{1}{2} D \int_A (\Delta w)^2 dA. \tag{2.58}$$

Ulkoisten kuormien potentiaali on

$$V = -\int_A p(x,y) w(x,y)\, dx\, dy - \int_{S_1} \left(-\bar{M}_n \frac{\partial w}{\partial n} + \bar{V}_n w \right) ds, \tag{2.59}$$

$$\bar{V}_n = \bar{Q}_n + \frac{\partial \bar{M}_{ns}}{\partial s}, \tag{2.60}$$

$$\Pi = U + V. \tag{2.61}$$

Potentiaalienergian minimin periaatteen mukaan todellinen taipuma antaa potentiaalienergialle minimiarvon tasapainotilassa hyväksyttävien taipumafunktioiden joukossa. Hyväksyttävä taipumafunktio on kahdesti jatkuvasti derivoituva ja toteuttaa kinemaattiset (geometriset) reunaehdot reunalla S_2

$$w = \bar{w}, \quad \frac{\partial w}{\partial n} = \frac{\partial \bar{w}}{\partial n} \quad \text{reunalla} \quad S_2. \tag{2.62}$$

2.2.3 Komplementaarisen energian minimin periaate

Laatan muodonmuutosenergia momenttien avulla

$$\bar{U} = \frac{1}{2D(1-\nu^2)} \int_A [(M_x + M_y)^2 - 2(1+\nu)(M_x M_y - M_{xy}^2)]\, dA. \tag{2.63}$$

Laatan ulkoinen komplementaarinen energia

$$\bar{V} = \int_{S_2} (\bar{w} V_n - \frac{\partial \bar{w}}{\partial n} M_n)\, ds, \tag{2.64}$$

viiva taipuman ja kaltevuuskulman päällä tarkoittaa reunalla S_2 annettua (tunnettua) arvoa. Komplementaarinen kokonaisenergia

$$\bar{\Pi} = \bar{U} + \bar{V}. \tag{2.65}$$

Todelliset momentit antavat komplementaariselle energialle minimiarvon hyväksyttävien (staattisesti luvallisten) momenttien joukossa, jonka jäsenet toteuttavat laatan tasapainoehdon

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x,y) = 0 \tag{2.66}$$

ja mekaaniset reunaehdot reunalla S_1 eli

$$M_n = \bar{M}_n, \quad V_n = \bar{V}_n, \tag{2.67}$$

$$V_n = Q_n + \frac{\partial M_{ns}}{\partial s}. \tag{2.68}$$

2.3 Levy

Hooken laki tasojännitystilassa

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \tag{2.69}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \tag{2.70}$$

$$\tau_{xy} = G\gamma_{xy}. \tag{2.71}$$

Tasapainoyhtälöt tasossa

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0, \tag{2.72}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + f_y = 0. \tag{2.73}$$

Jännityskomponentit Airyn jännitysfunktion avulla

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} + V, \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} + V, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}. \tag{2.74}$$

Yhteensopivuusehto

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2.75}$$

$$\Rightarrow \Delta\Delta\Phi + (1-\nu)\Delta V = 0, \tag{2.76}$$

$$\text{Laplace} \quad \Delta(\bullet) = \frac{\partial^2(\bullet)}{\partial x^2} + \frac{\partial^2(\bullet)}{\partial y^2}. \tag{2.77}$$

Tapauksessa $f_x = f_y =$ vakio, eli $\Delta V = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi(x,y) = 0, \tag{2.78}$$

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0. \tag{2.79}$$

Tasomuodonmuutostila

Tasomuodonmuutostilassa (x, y)–tasossa

$$\sigma_x = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}(\varepsilon_x + \frac{\nu}{1-\nu}\varepsilon_y), \tag{2.80}$$

$$\sigma_y = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}(\varepsilon_y + \frac{\nu}{1-\nu}\varepsilon_x), \tag{2.81}$$

$$\tau_{xy} = G\gamma_{xy}. \tag{2.82}$$

Voimamenetelmän biharmoninen differentiaaliyhtälö

$$\Delta\Delta\Phi + \frac{1-2\nu}{1-\nu}\Delta V = 0. \tag{2.83}$$

Reunaehdot

Reuna-alkion tasapainoehdot

$$t_x = \sigma_x \cos\alpha + \tau_{xy}\sin\alpha = \sigma_x n_x + \tau_{xy} n_y, \tag{2.84}$$

$$t_y = \tau_{xy}\cos\alpha + \sigma_y \sin\alpha = \tau_{xy} n_x + \sigma_y n_y, \tag{2.85}$$

$$n_x = \cos\alpha = \frac{dy}{ds}, \quad n_y = \sin\alpha = -\frac{dx}{ds} \tag{2.86}$$

ovat reunakäyrän normaalivektorin komponentit. Merkitsemällä

$$Q_x = \int_0^s t_x(s)\, ds \text{ ja } B = \left.\frac{\partial\Phi}{\partial y}\right|_{s=0}, \tag{2.87}$$

$$Q_y = \int_0^s t_y(s)\, ds \text{ ja } A = \left.\frac{\partial\Phi}{\partial x}\right|_{s=0}, \tag{2.88}$$

$$\Phi = \int_0^s \left[Q_y\left(-\frac{dx}{ds}\right) + Q_x\frac{dy}{ds}\right] ds + Ax + By + C. \tag{2.89}$$

Asettamalla $A = B = C = 0$

$$\Phi = \int_0^s (Q_y \sin\alpha + Q_x \cos\alpha)\, ds. \tag{2.90}$$

Jännitysfunktion derivaatta normaalin suuntaan

$$\frac{d\Phi}{dn} = \frac{\partial\Phi}{\partial x}\frac{dx}{dn} + \frac{\partial\Phi}{\partial y}\frac{dy}{dn} = -Q_y\cos\alpha + Q_x \sin\alpha. \tag{2.91}$$

Kehäanalogia

Kehäksi ajatellun reunakäyrän normaalivoima, leikkausvoima ja momentti

$$\frac{d\Phi}{dn} = N, \quad \frac{d\Phi}{ds} = Q, \quad \Phi(s) = M(s). \tag{2.92}$$

Siirtymien määrittäminen

$$\frac{\partial u}{\partial x} = \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \tag{2.93}$$

$$\frac{\partial v}{\partial y} = \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \tag{2.94}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} = \frac{1}{G}\tau_{xy}. \tag{2.95}$$

$$\begin{aligned} Eu =& E\int_a^x \varepsilon_x dx + \bar{f}_1(y) = \int_a^x (\sigma_x - \nu\sigma_y)\, dx + \bar{f}_1(y) \\ =& \int_a^x \frac{\partial^2\Phi}{\partial y^2}\, dx - \nu\frac{\partial\Phi}{\partial x} + f_1(y), \end{aligned} \tag{2.96}$$

$$\begin{aligned} Ev =& E\int_b^y \varepsilon_y dy + \bar{f}_2(x) = \int_b^y (\sigma_y - \nu\sigma_x)\, dy + \bar{f}_2(x) \\ =& \int_b^y \frac{\partial^2\Phi}{\partial x^2}\, dy - \nu\frac{\partial\Phi}{\partial y} + f_2(x). \end{aligned} \tag{2.97}$$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2\Phi}{\partial x\partial y} \Rightarrow \tag{2.98}$$

$$\int_a^x \frac{\partial^3\Phi}{\partial y^3}\, dx + \int_b^y \frac{\partial^3\Phi}{\partial x^3}\, dy + 2\frac{\partial^2\Phi}{\partial x\partial y} = -\frac{\partial f_1}{dy} - \frac{df_2}{dx} \equiv F_1(y) + F_2(x), \tag{2.99}$$

$$F_2(x) + \frac{df_2}{dx} = -F_1(y) - \frac{df_1}{dy} = \text{vakio} = c, \tag{2.100}$$

$$f_2(x) = -\int_a^x F_2(x)dx + cx + d_2, \quad f_1(y) = -\int_b^y F_2(y)dy + cy + d_1, \tag{2.101}$$

c, d_1 ja d_2 ratkaistaan reunaehdoista.

2.4 Fourier-sarjaratkaisut

Jaksollisen funktion Fourier-sarja välillä $(-L, L)$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x] \tag{2.102}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\, dx, \quad n = 0, 1, 2, 3, \ldots, \tag{2.103}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\, dx, \quad n = 1, 2, 3, \ldots. \tag{2.104}$$

Parillisen funktion Fourier–sarja

$$f(-x) = f(x) \quad \forall x, \tag{2.105}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, \ldots, \quad b_n = 0. \tag{2.106}$$

Parittoman funktion Fourier–sarja

$$f(-x) = -f(x), \tag{2.107}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots, \quad a_n = 0. \tag{2.108}$$

2.4.1 Puolitaso, jaksollinen reunakuorma

$$p(x) = \sum_{n=1}^{\infty} a_n \cos\alpha_n x, \quad \alpha_n = \frac{n\pi}{L}. \tag{2.109}$$

Jännitysfunktio

$$\Phi(x, y) = \sum_{n=1}^{\infty} Y_n(y)\cos\alpha_n x, \tag{2.110}$$

$$\Delta\Delta\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0 \tag{2.111}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\alpha_n^4 Y_n(y) - 2\alpha_n^2\frac{d^2Y(y)}{dy^2} + \frac{d^4Y_n(y)}{dy^4}\right]\cos\alpha_n x = 0, \tag{2.112}$$

$$Y_n(y) = (A_n + B_n\alpha_n y)e^{\alpha_n y} + (C_n + D_n\alpha_n y)e^{-\alpha_n y}, \tag{2.113}$$

$$\frac{dY_n}{dy} = \alpha_n[(A_n + B_n + B_n\alpha_n y)e^{\alpha_n y} + (-C_n + D_n - D_n\alpha_n y)e^{-\alpha_n y}], \tag{2.114}$$

$$\frac{d^2Y_n}{dy^2} = \alpha_n^2[(A_n + 2B_n + B_n\alpha_n y)e^{\alpha_n y} + (C_n - 2D_n + D_n\alpha_n y)e^{-\alpha_n y}], \tag{2.115}$$

$$\text{tai} \quad Y_n(y) = (A_n + B_n\alpha_n y)\cosh\alpha_n y + (C_n + D_n\alpha_n y)\sinh\alpha_n y. \tag{2.116}$$

$$\frac{dY_n}{dy} = \alpha_n[(A_n + D_n + B_n\alpha_n y)\sinh\alpha_n y + (C_n + B_n + D_n\alpha_n y)\cosh\alpha_n y]. \tag{2.117}$$

$$\frac{d^2Y_n}{dy^2} = \alpha_n^2[(A_n + 2D_n + B_n\alpha_n y)\cosh\alpha_n y + (C_n + 2B_n + D_n\alpha_n y)\sinh\alpha_n y]. \tag{2.118}$$

Jännityskomponentit

$$\sigma_y = \frac{\partial^2\Phi(x, y)}{\partial x^2}, \quad \sigma_x = \frac{\partial^2\Phi(x, y)}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2\Phi(x, y)}{\partial x\partial y}. \tag{2.119}$$

2.4.2 Levykaista, jaksollinen reunakuorma

$$p(x) = \sum_{n=1}^{\infty} a_n \cos \alpha_n x, \quad \alpha_n = \frac{n\pi}{L}. \tag{2.120}$$

Jännitysfunktio

$$\Phi(x, y) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} [(A_n + B_n \alpha_n y) \cosh \alpha_n y + (C_n + D_n \alpha_n y) \sinh \alpha_n y] \cos \alpha_n x. \tag{2.121}$$

Jännitykset

$$\sigma_y(x, y) = -\sum_{n=1}^{\infty} [(A_n + B_n \alpha_n y) \cosh \alpha_n y + (C_n + D_n \alpha_n y) \sinh \alpha_n y] \cos \alpha_n x, \tag{2.122}$$

$$\sigma_x(x, y) = \sum_{n=1}^{\infty} [(A_n + 2D_n + B_n \alpha_n y) \cosh \alpha_n y + (C_n + 2B_n + D_n \alpha_n y) \sinh \alpha_n y] \cos \alpha_n x, \tag{2.123}$$

$$\tau_{xy}(x, y) = \sum_{n=1}^{\infty} [(A_n + D_n + B_n \alpha_n y) \sinh \alpha_n y + (C_n + B_n + D_n \alpha_n y) \cosh \alpha_n y] \sin \alpha_n x, \tag{2.124}$$

2.5 Fourier-muunnos -ratkaisut

Parittoman funktion, $f(x) = -f(-x)$ Fourier-sinimuunnos ja käänteismuunnos

$$\bar{f}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\xi) \sin \alpha\xi \, d\xi, \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}(\alpha) \sin \alpha x \, d\alpha. \tag{2.125}$$

Parillisen funktion $f(-x) = f(x)$ kosinimuunnos

$$\bar{f}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\xi) \cos \alpha\xi \, d\xi, \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}(\alpha) \cos \alpha x \, d\alpha. \tag{2.126}$$

Yleisen tapauksen Fourier-muunnos

$$\bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{i\alpha\xi} \, d\xi \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-i\alpha x} \, d\alpha. \tag{2.127}$$

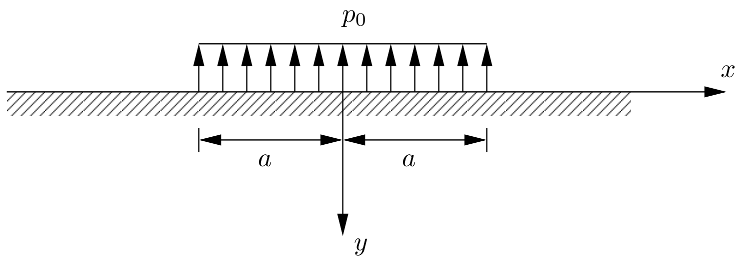
Kaksiulotteisessa tapauksessa funktion $f(x, y)$ kosinimuunnos

$$\bar{f}(\alpha, y) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x, y) \cos \alpha x \, dx, \quad f(x, y) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}(\alpha, y) \cos \alpha x \, d\alpha. \tag{2.128}$$

Funktion f toisen ja 4. derivaatan muunnos

$$\begin{aligned} \bar{f}_{,xx}(\alpha, y) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_{,xx}(x, y) \cos \alpha x \, dx \\ &= -\alpha^2 \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x, y) \cos \alpha x \, dx = -\alpha^2 \bar{f}(\alpha, y), \end{aligned} \tag{2.129}$$

$$\bar{f}_{,xxxx} = \alpha^4 \bar{f}. \tag{2.130}$$



Kuva 2.2 Leimakuorma.

2.5.1 Puolitason reunalla symmetrinen kuorma $p(x)$

$$\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0, \tag{2.131}$$

$$\begin{aligned}\sqrt{\frac{2}{\pi}}\int_0^\infty \Delta\Delta\Phi\cos\alpha x\,dx =& \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\alpha^4\Phi(x,y) - 2\alpha^2\frac{\partial^2\Phi(x,y)}{\partial y^2} + \frac{\partial^4\Phi(x,y)}{\partial y^4}\right]\cos\alpha x\,dx \\ =& \alpha^4\bar{\Phi}(\alpha,y) - 2\alpha^2\frac{d\bar{\Phi}(\alpha,y)}{dy^2} + \frac{d^4\bar{\Phi}(\alpha,y)}{dy^4} = 0.\end{aligned} \tag{2.132}$$

Jännitysfunktion ja jännityskomponenttien muunnokset

$$\bar{\Phi}(\alpha,y) = (A + B\alpha y)e^{-\alpha y} + (C + D\alpha y)e^{\alpha y}, \tag{2.133}$$

$$\frac{d\bar{\Phi}(\alpha,y)}{dy} = -\alpha(A - B + B\alpha y)e^{-\alpha y} + \alpha(C + D + D\alpha y)e^{\alpha y}, \tag{2.134}$$

$$\frac{d^2\bar{\Phi}(\alpha,y)}{dy^2} = \alpha^2(A - 2B + B\alpha y)e^{-\alpha y} + \alpha^2(C + 2D + D\alpha y)e^{\alpha y}, \tag{2.135}$$

$$\bar{\sigma}_y(\alpha,y) = -\alpha^2\bar{\Phi}(\alpha,y), \tag{2.136}$$

$$\bar{\tau}_{xy}(\alpha,y) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2\Phi}{\partial x\partial y}(x,y)\sin\alpha x\,dx = \alpha\frac{d\bar{\Phi}(\alpha,y)}{dy}. \tag{2.137}$$

Käänteismuunnoksella

$$\Phi(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{\Phi}(\alpha,y)\cos\alpha x\,d\alpha. \tag{2.138}$$

2.5.2 Erikoistapaus leimakuorma

Integraalien laskeminen

$$\int_0^\infty \frac{1}{\alpha}\sin\alpha a(1 - \alpha y)e^{-\alpha y}\cos\alpha x\,d\alpha \equiv [I_1 - I_2], \tag{2.139}$$

$$I_1 = \int_0^\infty \frac{1}{\alpha}\sin\alpha a\,e^{-\alpha y}\cos\alpha x\,d\alpha, \tag{2.140}$$

$$I_2 = \int_0^\infty \sin\alpha a\, y\, e^{-\alpha y} \cos\alpha x \, d\alpha, \tag{2.141}$$

$$\sin\alpha a \cos\alpha x = \frac{1}{2}\sin\alpha(x+a) - \frac{1}{2}\sin\alpha(x-a), \tag{2.142}$$

$$I_1 = \frac{1}{2}\int_0^\infty \frac{1}{\alpha}\sin\alpha(x+a)e^{-\alpha y}\, d\alpha - \frac{1}{2}\int_0^\infty \frac{1}{\alpha}\sin\alpha(x-a)e^{-\alpha y}\, d\alpha. \tag{2.143}$$

Soveltamalla integraalin laskemiseen liitteen 3 kaavaa

$$\int_0^\infty \frac{1}{\alpha}e^{-\alpha y}\, d\alpha = \arctan\frac{x}{y} \tag{2.144}$$

$$I_1 = \frac{1}{2}\arctan\frac{x+a}{y} - \frac{1}{2}\arctan\frac{x-a}{y}, \tag{2.145}$$

$$I_2 = \frac{1}{2}\int_0^\infty \sin\alpha(x+a)ye^{-\alpha y}\, d\alpha - \frac{1}{2}\int_0^\infty \sin\alpha(x-a)ye^{-\alpha y}\, d\alpha, \tag{2.146}$$

$$\int_0^\infty e^{-\alpha y}\sin\alpha x\, d\alpha = \frac{x}{x^2+y^2}, \quad (\Re y > 0), \tag{2.147}$$

$$I_2 = \frac{1}{2}y\left[\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right], \tag{2.148}$$

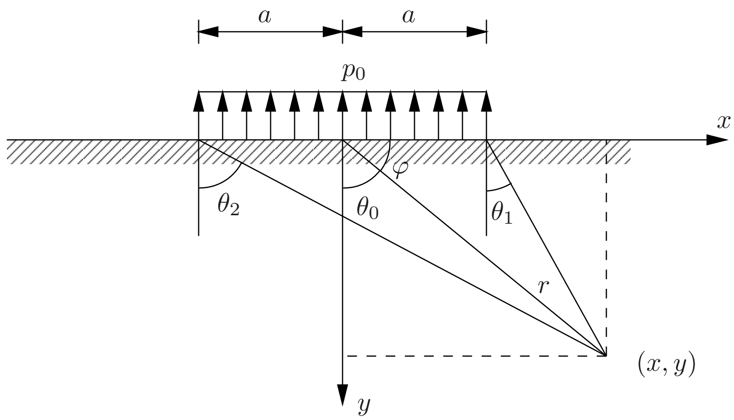
$$\sigma_x = \frac{p_0}{\pi}\left\{\arctan\frac{x+a}{y} - \arctan\frac{x-a}{y} - y\left[\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right]\right\}, \tag{2.149}$$

$$\sigma_y = \frac{p_0}{\pi}\left\{\arctan\frac{x+a}{y} - \arctan\frac{x-a}{y} + y\left[\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right]\right\}, \tag{2.150}$$

$$\sin\alpha a \sin\alpha x = \left[\frac{1}{2}\cos\alpha(x-a) - \frac{1}{2}\cos\alpha(x+a)\right] \tag{2.151}$$

$$\int_0^\infty e^{-\alpha y}\cos\alpha x\, d\alpha = \frac{y}{x^2+y^2}, \tag{2.152}$$

$$\tau_{xy} = \frac{p_0}{\pi}\left[\frac{y^2}{(x-a)^2+y^2} - \frac{y^2}{(x+a)^2+y^2}\right]. \tag{2.153}$$



Kuva 2.3 Kulma θ.

2.5.3 Pistekuorma F puolitason reunalla

$$\sigma_x = \frac{2F}{\pi}\frac{\cos\theta_0\sin^2\theta_0}{r} = \frac{2F}{\pi}\frac{\sin\varphi\cos^2\varphi}{r}, \tag{2.154}$$

$$\sigma_y = \frac{2F}{\pi}\frac{\cos^3\theta_0}{r} = \frac{2F}{\pi}\frac{\sin^3\varphi}{r}, \tag{2.155}$$

$$\tau_{xy} = \frac{2F}{\pi}\frac{\cos^2\theta_0\sin\theta_0}{r} = \frac{2F}{\pi}\frac{\sin^2\varphi\cos\varphi}{r}. \tag{2.156}$$

Siirtymällä napakoordinaatistoon tulee kaavojen

$$\sigma_r = \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi + 2\tau_{xy}\sin\varphi\cos\varphi, \tag{2.157}$$

$$\sigma_\varphi = \sigma_x\sin^2\varphi + \sigma_y\cos^2\varphi - 2\tau_{xy}\sin\varphi\cos\varphi, \tag{2.158}$$

$$\tau_{xy} = (\sigma_y - \sigma_x)\sin\varphi\cos\varphi + \tau_{xy}(\cos^2\varphi - \sin^2\varphi) \tag{2.159}$$

avulla

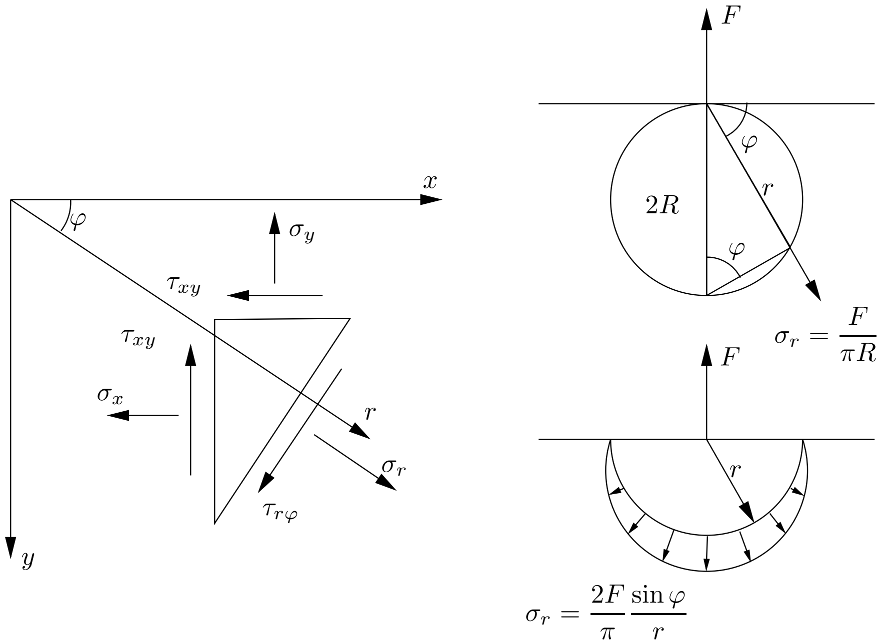
$$\sigma_r = \frac{2F}{\pi}\frac{\sin\varphi}{r}, \quad \sigma_\varphi = 0, \quad \tau_{r\varphi} = 0. \tag{2.160}$$

R-säteisellä ympyrällä

$$\sigma_r = \frac{F}{\pi R}. \tag{2.161}$$

Ottamalla huomioon

$$2R\sin\varphi = r \Rightarrow \sigma_r = \frac{2F}{\pi}\frac{\sin\varphi}{r} = \frac{2F}{\pi}\frac{1}{2R} = \frac{F}{\pi R}. \tag{2.162}$$



Kuva 2.4 Jännityskomponenttien muunnoksia.

2.5.4 Puolitason reunalla y–akselin suhteen symmetrinen leikkauskuorma $q(x)$

Jännitysfunktio $\Phi(x, y)$ antisymmetrinen y–akselin suhteen. Sinimuunnoksella

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \left[\frac{\partial^4\Phi(x,y)}{\partial x^4} + 2\frac{\partial^4\Phi(x,y)}{\partial x^2\partial y^2} + \frac{\partial^4\Phi(x,y)}{\partial y^4}\right]\sin\alpha x\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \left[\alpha^4\Phi(x,y) - 2\alpha^2\frac{\partial^2\Phi(x,y)}{\partial y^2} + \frac{\partial^4\Phi(x,y)}{\partial y^4}\right]\sin\alpha x\,dx \tag{2.163}$$

$$=\alpha^4\bar{\Phi}(\alpha,y) - 2\alpha^2\frac{d\bar{\Phi}(\alpha,y)}{dy^2} + \frac{d^4\bar{\Phi}(\alpha,y)}{dy^4} = 0,$$

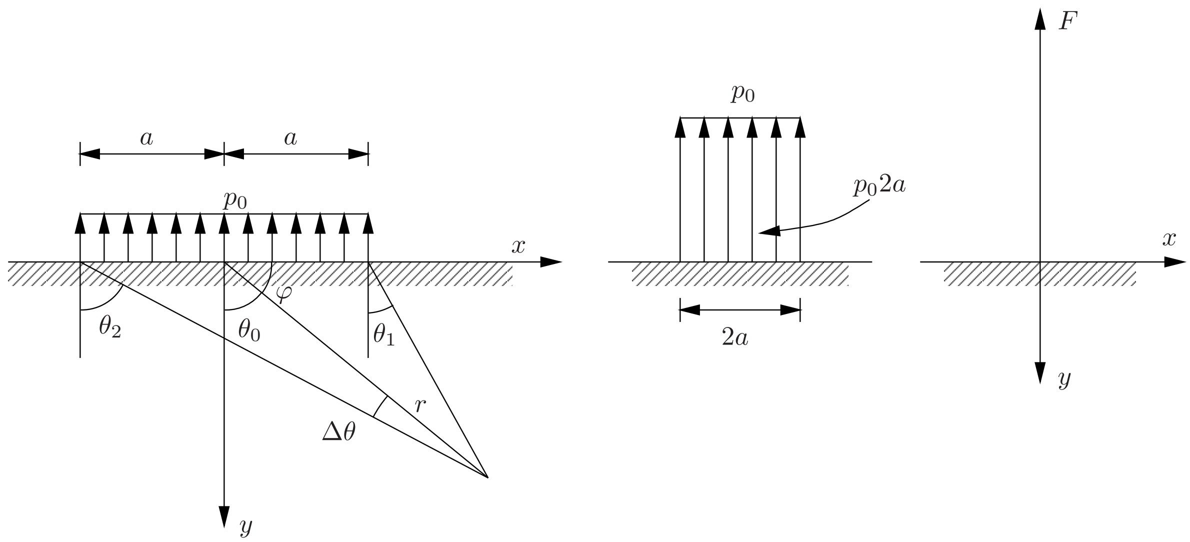
$$\bar{\Phi}(\alpha,y) = (A + B\alpha y)e^{-\alpha y} + (C + D\alpha y)e^{\alpha y}, \tag{2.164}$$

$$\bar{\sigma}_y(\alpha,y) = -\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi(x,y)\sin\alpha x\,dx = -\alpha^2\bar{\Phi}(\alpha,y), \tag{2.165}$$

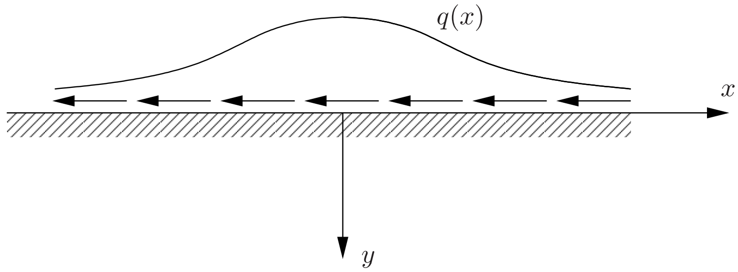
$$\bar{\tau}_{xy}(\alpha,y) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2\Phi}{\partial x\partial y}(x,y)\cos\alpha x\,dx = -\alpha\frac{d\Phi(\alpha,y)}{dy}, \tag{2.166}$$

$$\bar{q}(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty q(x)\cos\alpha x\,dx, \tag{2.167}$$

$$\int_0^\infty |q(x)|\,dx < 0, \tag{2.168}$$



Kuva 2.5 Pistekuorma puolitason reunalla.



Kuva 2.6 Symmetrinen leikkauskuorma.

$$\Phi(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{\Phi}(\alpha, y) \sin \alpha x \, d\alpha. \tag{2.169}$$

2.5.5 Erikoistapaus $q = q_0$, kun $\mid x \mid < a$

$$\sigma_x = \frac{q_0}{\pi} \left\{ \ln \sqrt{\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}} - y^2 \left[\frac{1}{(x-a)^2 + y^2} - \frac{1}{(x+a)^2 + y^2} \right] \right\}, \tag{2.170}$$

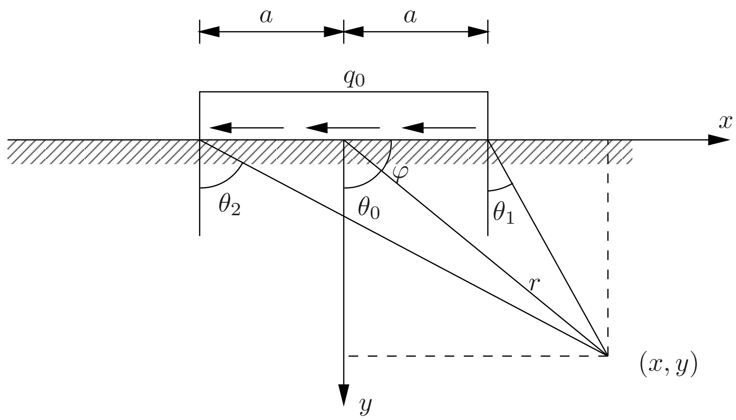
$$\sigma_y = \frac{q_0}{\pi} y^2 \left[\frac{1}{(x-a)^2 + y^2} - \frac{1}{(x+a)^2 + y^2} \right], \tag{2.171}$$

$$\tau_{xy} = \frac{q_0}{\pi} \left\{ \arctan \frac{x+a}{y} - \arctan \frac{x-a}{y} - y \left[\frac{x+a}{(x+a)^2 + y^2} - \frac{x-a}{(x-a)^2 + y^2} \right] \right\}. \tag{2.172}$$

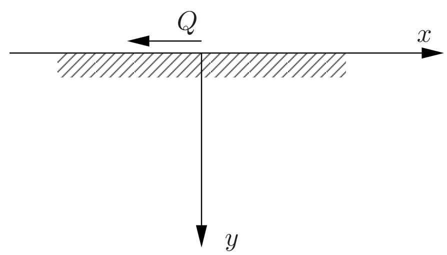
2.5.6 Pistekuorma Q puolitason reunalla

$$\sigma_x = \frac{2Q}{\pi} \frac{\sin^3 \theta_0}{r} = \frac{2Q}{\pi} \frac{\cos^3 \varphi}{r}, \tag{2.173}$$

$$\sigma_y = \frac{2Q}{\pi} \frac{\cos^2 \theta_0 \sin \theta_0}{r} = \frac{2Q}{\pi} \frac{\sin^2 \varphi \cos \varphi}{r}, \tag{2.174}$$



Kuva 2.7 Tasainen leikkauskuorma.



Kuva 2.8 Pistemäinen leikkauskuorma.

$$\tau_{xy} = \frac{2Q}{\pi}\frac{\cos\theta_0 \sin^2\theta_0}{r} = \frac{2Q}{\pi}\frac{\sin\varphi\cos^2\varphi}{r}. \tag{2.175}$$

Kaavojen

$$\sigma_r = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi + 2\tau_{xy}\sin\varphi\cos\varphi, \tag{2.176}$$

$$\sigma_\varphi = \sigma_x \sin^2\varphi + \sigma_y \cos^2\varphi - 2\tau_{xy}\sin\varphi\cos\varphi, \tag{2.177}$$

$$\tau_{xy} = (\sigma_y - \sigma_x)\sin\varphi\cos\varphi + \tau_{xy}(\cos^2\varphi - \sin^2\varphi) \tag{2.178}$$

avulla

$$\sigma_r = \frac{2Q}{\pi}\frac{\cos\varphi}{r}, \quad \sigma_\varphi = 0, \quad \tau_{r\varphi} = 0. \tag{2.179}$$

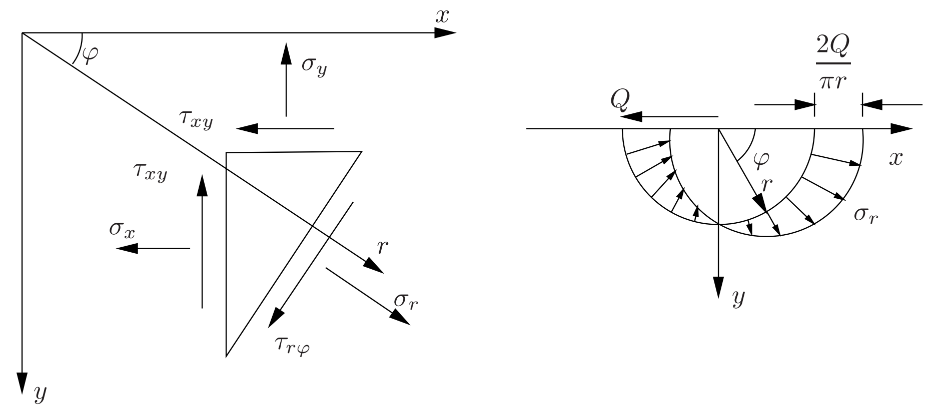
2.6 Levy polaarikoordinaatistossa

$$\nabla^2\Phi(x,y) = \frac{\partial^2\Phi(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi(r,\varphi)}{\partial\varphi^2}, \tag{2.180}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)\left[\frac{\partial^2\Phi(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi(r,\varphi)}{\partial\varphi^2}\right] = 0, \tag{2.181}$$

$$\sigma_r(r,\varphi) = \frac{1}{r}\frac{\partial\Phi(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi(r,\varphi)}{\partial\varphi^2} + V(r,\varphi), \tag{2.182}$$

$$\sigma_\varphi(r,\varphi) = \frac{\partial^2\Phi(r,\varphi)}{\partial r^2} + V(r,\varphi), \tag{2.183}$$



Kuva 2.9 Vaakasuuntaisen pistevoiman Q aiheuttama jännitystila ympyräviivalla $r =$ vakio.

$$\begin{aligned}\tau_{r\varphi}(r,\varphi) =& \frac{1}{r^2}\frac{\partial \Phi(r,\varphi)}{\partial \varphi} - \frac{1}{r}\frac{\partial^2 \Phi(r,\varphi)}{\partial r \partial \varphi} \\ =& -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial \Phi(r,\varphi)}{\partial \varphi}\right].\end{aligned} \tag{2.184}$$

Muodonmuutokset

$$u = u(r,\varphi), \quad v = v(r,\varphi), \tag{2.185}$$

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\varphi = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \varphi}, \tag{2.186}$$

$$\gamma_{r\varphi} = \frac{1}{r}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} - \frac{v}{r}. \tag{2.187}$$

Tasojännitystilassa

$$\begin{aligned}\sigma_r =& \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\varphi), \\ \sigma_\varphi =& \frac{E}{1-\nu^2}(\varepsilon_\varphi + \nu\varepsilon_r), \\ \tau_{r\varphi} =& G\gamma_{r\varphi}, \quad G = \frac{E}{2(1+\nu)}.\end{aligned} \tag{2.188}$$

Tasapainoyhtälöt

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\varphi r}}{\partial \varphi} + \frac{\sigma_r - \sigma_\varphi}{r} + f_r = 0, \tag{2.189}$$

$$\frac{1}{r}\frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{\partial \tau_{r\varphi}}{\partial r} + \frac{2\tau_{\varphi r}}{r} + f_\varphi = 0. \tag{2.190}$$

Pyörähdyssymmetrinen jännitystila

$$\nabla^4\Phi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left[\frac{d^2\Phi(r)}{dr^2} + \frac{1}{r}\frac{d\Phi(r)}{dr}\right] = 0, \tag{2.191}$$

$$\frac{d^4\Phi}{dr^4} + \frac{2}{r}\frac{d^3\Phi}{dr^3} - \frac{1}{r}^2\frac{d^2\Phi}{dr^2} + \frac{1}{r}^3\frac{d\Phi}{dr} = 0, \tag{2.192}$$

$$\sigma_r = \frac{1}{r}\frac{d\Phi}{dr}, \quad \sigma_\varphi(r) = \frac{d^2\Phi(r)}{dr^2}. \tag{2.193}$$

Identiteetti

$$\nabla^2\Phi(r) = \frac{d^2\Phi(r)}{dr^2} + \frac{1}{r}\frac{d\Phi(r)}{dr} = \frac{1}{r}\frac{d}{dr}\left[r\frac{d\Phi(r)}{dr}\right], \tag{2.194}$$

jännitysfunktion differentiaaliyhtälö

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\Phi(r)}{dr}\right]\right\}\right) = 0, \tag{2.195}$$

$$\Phi(r) = A\ln r + Br^2\ln r + Cr^2 + D, \tag{2.196}$$

$$\sigma_r = \frac{A}{r^2} + B(1 + 2\ln r) + 2C, \tag{2.197}$$

$$\sigma_\varphi = -\frac{A}{r} + B(3 + 2\ln r) + 2C. \tag{2.198}$$

Siirtymämenetelmä pyörähdyssymmetrisessä muodonmuutostilassa

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right), \tag{2.199}$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right), \tag{2.200}$$

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} + f_r = 0 \tag{2.201}$$

$$\Rightarrow \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \frac{1-\nu^2}{E}f_r = 0, \tag{2.202}$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] + \frac{1-\nu^2}{E}f_r = 0, \tag{2.203}$$

f_r on tilavuusvoiman komponentti, esim. keskipakoisvoima

$$f_r = \rho\omega^2 r. \tag{2.204}$$

Ympyrärengas

Säde a

$$\varepsilon_\varphi(a) = \frac{u}{a} + \frac{1}{a}\frac{\partial v}{\partial\varphi}, \quad \gamma_{r\varphi}(a) = \frac{1}{a}\frac{\partial u}{\partial\varphi} + \frac{\partial v}{\partial a} - \frac{v}{a}. \tag{2.205}$$

Pyörähdyssymmetrisessä tapauksessa

$$\varepsilon_\varphi(a) = \frac{u}{a}, \quad \varepsilon_r = 0, \quad \gamma_{r\varphi} = 0. \tag{2.206}$$

2.7 Kuori

2.7.1 Kuoren geometria

Kaarialkion neliö

$$ds^2 = |d\mathbf{r}|^2 = d\mathbf{r} \cdot d\mathbf{r} = Ed\alpha^2 + 2Fd\alpha d\beta + Gd\beta^2, \tag{2.207}$$

$$E = \frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \alpha} = \frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \alpha} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \alpha} + \frac{\partial z}{\partial \alpha}\frac{\partial z}{\partial \alpha}, \tag{2.208}$$

$$F = \frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \beta} = \frac{\partial x}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \beta} + \frac{\partial z}{\partial \alpha}\frac{\partial z}{\partial \beta}, \tag{2.209}$$

$$G = \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial \mathbf{r}}{\partial \beta} = \frac{\partial x}{\partial \beta}\frac{\partial x}{\partial \beta} + \frac{\partial y}{\partial \beta}\frac{\partial y}{\partial \beta} + \frac{\partial z}{\partial \beta}\frac{\partial z}{\partial \beta}. \tag{2.210}$$

Tangenttivektorit

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{r}}{\partial \alpha}, \quad \mathbf{g}_\beta = \frac{\partial \mathbf{r}}{\partial \beta}, \tag{2.211}$$

$$F = \frac{\partial \mathbf{r}}{\partial \alpha} \cdot \frac{\partial \mathbf{r}}{\partial \beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = \sqrt{EG}\cos\chi. \tag{2.212}$$

$$ds^2 = ds_\alpha^2 + ds_\beta^2 = A^2 d\alpha^2 + B^2 d\beta^2, \tag{2.213}$$

$$A = \sqrt{E} \quad \text{ja} \quad B = \sqrt{G}. \tag{2.214}$$

Pinnan normaalivektori

$$\mathbf{n} = \frac{\mathbf{g}_\alpha \times \mathbf{g}_\beta}{|\mathbf{g}_\alpha \times \mathbf{g}_\beta|} = \frac{1}{H}\left(\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial \beta}\right), \tag{2.215}$$

$$H = \left|\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial \beta}\right| = \left|\frac{\partial \mathbf{r}}{\partial \alpha}\right|\left|\frac{\partial \mathbf{r}}{\partial \beta}\right| \sin\chi = \sqrt{EG}\sin\chi, \tag{2.216}$$

$$\sin\chi = \sqrt{1-\cos^2\chi} = \sqrt{1-\frac{F^2}{EG}}, \tag{2.217}$$

$$H = \sqrt{EG - F^2}. \tag{2.218}$$

Pinnan toinen neliömuoto

$$d\mathbf{r} \cdot d\mathbf{n} = Ld\alpha^2 + 2Md\alpha d\beta + Nd\beta^2, \tag{2.219}$$

$$L = \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial \alpha^2} = \frac{1}{H}\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial^2 \mathbf{r}}{\partial \alpha^2} = \frac{1}{H}\begin{vmatrix} \frac{\partial^2 x}{\partial \alpha^2} & \frac{\partial^2 y}{\partial \alpha^2} & \frac{\partial^2 z}{\partial \alpha^2} \\ \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \end{vmatrix}, \tag{2.220}$$

$$M = \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial \alpha \partial \beta} = \frac{1}{H}\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial^2 \mathbf{r}}{\partial \alpha \partial \beta} = \frac{1}{H}\begin{vmatrix} \frac{\partial^2 x}{\partial \alpha \partial \beta} & \frac{\partial^2 y}{\partial \alpha \partial \beta} & \frac{\partial^2 z}{\partial \alpha \partial \beta} \\ \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \end{vmatrix}, \tag{2.221}$$

$$N = \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial \beta^2} = \frac{1}{H} \frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial \beta} \cdot \frac{\partial^2 \mathbf{r}}{\partial \beta^2} = \frac{1}{H} \begin{vmatrix} \frac{\partial^2 x}{\partial \beta^2} & \frac{\partial^2 y}{\partial \beta^2} & \frac{\partial^2 z}{\partial \beta^2} \\ \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial \beta} & \frac{\partial y}{\partial \beta} & \frac{\partial z}{\partial \beta} \end{vmatrix}. \tag{2.222}$$

$$\frac{\partial \mathbf{n}}{\partial \alpha} = \frac{FM - GL}{H^2} \frac{\partial \mathbf{r}}{\partial \alpha} + \frac{FL - EM}{H^2} \frac{\partial \mathbf{r}}{\partial \beta}, \tag{2.223}$$

$$\frac{\partial \mathbf{n}}{\partial \beta} = \frac{FN - GM}{H^2} \frac{\partial \mathbf{r}}{\partial \alpha} + \frac{FM - EN}{H^2} \frac{\partial \mathbf{r}}{\partial \beta}. \tag{2.224}$$

Kaarevuusviivojen suunnat:

$$(EM - FL) \left(\frac{d\alpha}{d\beta} \right)^2 + (EN - GL) \frac{d\alpha}{d\beta} + (FN - GM) = 0. \tag{2.225}$$

Pääkaarevuudet:

$$(kE - L)(kG - N) - (kF - M)^2 = 0, \tag{2.226}$$

juuret k_1 ja k_2 ovat pääkaarevuudet. Jos α– ja β–viivat ovat kaarevuusviivat, niin $F = M = 0$ ja

$$k_1 = \frac{1}{R_1} = \frac{L}{E} = \frac{L}{A^2}, \tag{2.227}$$

$$k_2 = \frac{1}{R_2} = \frac{N}{G} = \frac{N}{B^2}. \tag{2.228}$$

Tässä tapauksessa ovat voimassa Gaussin-Godazzin kaavat

$$\frac{\partial}{\partial \alpha}(k_2 B) = k_1 \frac{\partial B}{\partial \alpha}, \quad \frac{\partial}{\partial \beta}(k_1 A) = k_2 \frac{\partial A}{\partial \beta}, \tag{2.229}$$

$$\frac{\partial}{\partial \alpha} \left(\frac{1}{A} \frac{\partial B}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{1}{B} \frac{\partial A}{\partial \beta} \right) = -k_1 k_2 AB. \tag{2.230}$$

Pinnan Gaussin kaarevuus

$$k_G = k_1 k_2 = \frac{1}{R_1 R_2}, \tag{2.231}$$

pinnan keskikaarevuus

$$k_m = \frac{1}{2}(k_1 + k_2) = \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right). \tag{2.232}$$

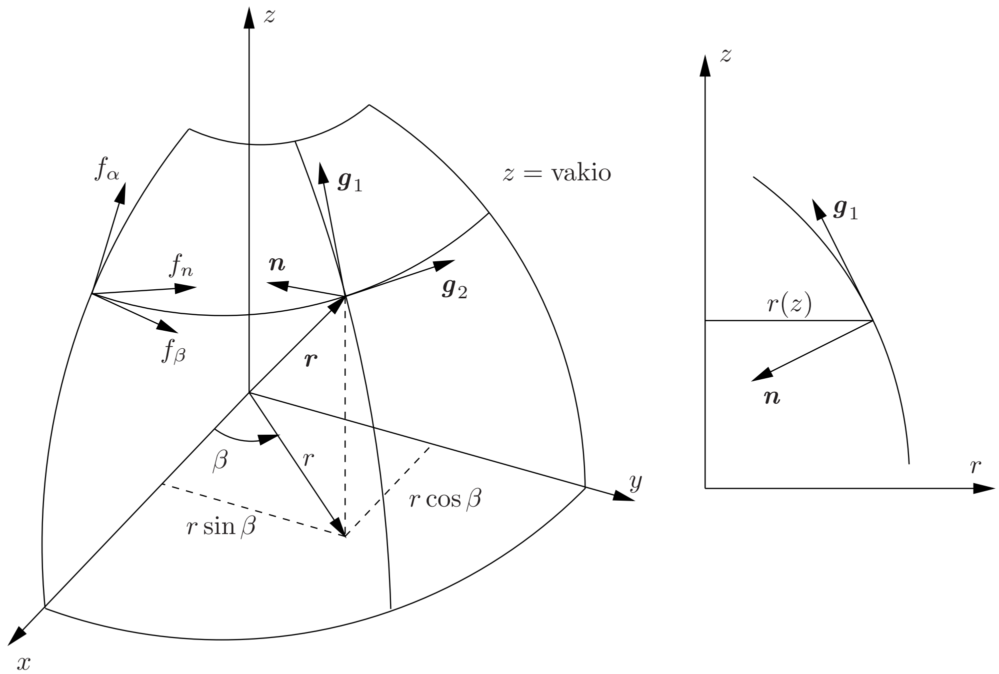
2.7.2 Kalvotilan tasapainoyhtälöt

$$\frac{\partial}{\partial \alpha}(BN_\alpha) - \frac{\partial B}{\partial \alpha} N_\beta + \frac{\partial}{\partial \beta}(AN_{\beta\alpha}) + \frac{\partial A}{\partial \beta} N_{\alpha\beta} + ABf_\alpha = 0, \tag{2.233}$$

$$\frac{\partial}{\partial \beta}(AN_\beta) - \frac{\partial A}{\partial \beta} N_\alpha + \frac{\partial}{\partial \alpha}(BN_{\alpha\beta}) + \frac{\partial B}{\partial \alpha} N_{\beta\alpha} + ABf_\beta = 0, \tag{2.234}$$

$$\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} - f_n = 0, \tag{2.235}$$

$$N_{\alpha\beta} - N_{\beta\alpha} = 0. \tag{2.236}$$



Kuva 2.10 Pyörähdyskuori.

2.7.3 Pyörähdyskuoren kalvotila

$$r = r(z), \tag{2.237}$$

$$x = r(z)\cos\beta, \quad y = r(z)\sin\beta, \tag{2.238}$$

$$\mathbf{r} = r(z)\cos\beta\mathbf{i} + r(z)\sin\beta\mathbf{j} + z\mathbf{k}, \tag{2.239}$$

$$\frac{\partial\mathbf{r}}{\partial z} = \frac{\partial r(z)}{\partial z}\sin\beta\mathbf{i} + \frac{\partial r(z)}{\partial z}\cos\beta\mathbf{j} + \mathbf{k}, \tag{2.240}$$

$$\frac{\partial\mathbf{r}}{\partial\beta} = -r(z)\sin\beta\mathbf{i} + r(z)\cos\beta\mathbf{j}. \tag{2.241}$$

Pinnan 1. neliömuodon kertoimet

$$A^2 = \frac{\partial\mathbf{r}}{\partial z}\cdot\frac{\partial\mathbf{r}}{\partial z} = \left(\frac{\partial r}{\partial z}\right)^2\cos^2\beta + \left(\frac{\partial r}{\partial z}\right)^2\sin^2\beta + 1 = 1 + \left(\frac{\partial r}{\partial z}\right)^2, \tag{2.242}$$

$$B^2 = \frac{\partial\mathbf{r}}{\partial\beta}\cdot\frac{\partial\mathbf{r}}{\partial\beta} = r^2\sin^2\beta + r^2\cos^2\beta = r^2, \tag{2.243}$$

$$A = \sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}, \quad B = r, \quad H = AB = r\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}, \tag{2.244}$$

$$F = \frac{\partial\mathbf{r}}{\partial\beta}\cdot\frac{\partial\mathbf{r}}{\partial z} = 0, \tag{2.245}$$

$$\mathbf{n} = \frac{1}{H}\left|\frac{\partial \mathbf{r}}{\partial \alpha} \times \frac{\partial \mathbf{r}}{\partial \beta}\right| = \frac{1}{H}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial r}{\partial z}\cos\beta & \frac{\partial r}{\partial z}\sin\beta & 1 \\ -r\sin\beta & r\cos\beta & 0 \end{vmatrix} \tag{2.246}$$

$$= -\frac{r}{H}(\cos\beta\mathbf{i} + \sin\beta\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}),$$

$$\frac{\partial^2 \mathbf{r}}{\partial z^2} = \frac{\partial^2 r(z)}{\partial z^2}\cos\beta\mathbf{i} + \frac{\partial^2 r(z)}{\partial z^2}\sin\beta\mathbf{j}, \tag{2.247}$$

$$\frac{\partial^2 \mathbf{r}}{\partial \beta^2} = -r(z)\cos\beta\mathbf{i} - r(z)\sin\beta\mathbf{j}. \tag{2.248}$$

Pinnan toisen neliömuodon kertoimet L, N, ja M

$$L = -\mathbf{n}\cdot\frac{\partial^2 \mathbf{r}}{\partial z^2} = -\frac{r}{H}\left(\frac{\partial^2 r}{\partial z^2}\cos^2\beta + \frac{\partial^2 r}{\partial z^2}\sin^2\beta\right) = -\frac{1}{H}r\frac{\partial^2 r}{\partial z^2}, \tag{2.249}$$

$$N = -\mathbf{n}\cdot\frac{\partial^2 \mathbf{r}}{\partial \beta^2} = \frac{1}{H}(r^2\cos^2\beta + r^2\sin^2\beta) = \frac{r^2}{H}, \tag{2.250}$$

$$M = -\mathbf{n}\cdot\frac{\partial^2 \mathbf{r}}{\partial z\partial \beta} = 0, \tag{2.251}$$

$$\frac{1}{R_\alpha} = \frac{L}{A^2} = -\frac{\frac{\partial^2 r}{\partial z^2}}{\left[1 + \left(\frac{\partial r}{\partial z}\right)^2\right]^{\frac{3}{2}}}, \quad \frac{1}{R_\beta} = \frac{N}{B^2} = \frac{1}{r\left[1 + \left(\frac{\partial r}{\partial z}\right)^2\right]^{\frac{1}{2}}}. \tag{2.252}$$

Tasapainoyhtälöt

$$\frac{\partial}{\partial z}(rN_\alpha) - \frac{\partial r}{\partial z}N_\beta + \sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}\,\frac{\partial N_{\alpha\beta}}{\partial \beta} + r\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}\,f_\alpha = 0, \tag{2.253}$$

$$\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}\,\frac{\partial N_\beta}{\partial \beta} + \frac{1}{r}\frac{\partial}{\partial z}(r^2 N_{\alpha\beta}) + r\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}\,f_\beta = 0, \tag{2.254}$$

$$-\frac{r\frac{\partial^2 r}{\partial z^2}}{1 + \left(\frac{\partial r}{\partial z}\right)^2}N_\alpha + N_\beta - r\sqrt{1 + \left(\frac{\partial r}{\partial z}\right)^2}\,f_n = 0. \tag{2.255}$$

Homogeenisen probleeman

$$f_\alpha = f_\beta = f_n = 0 \tag{2.256}$$

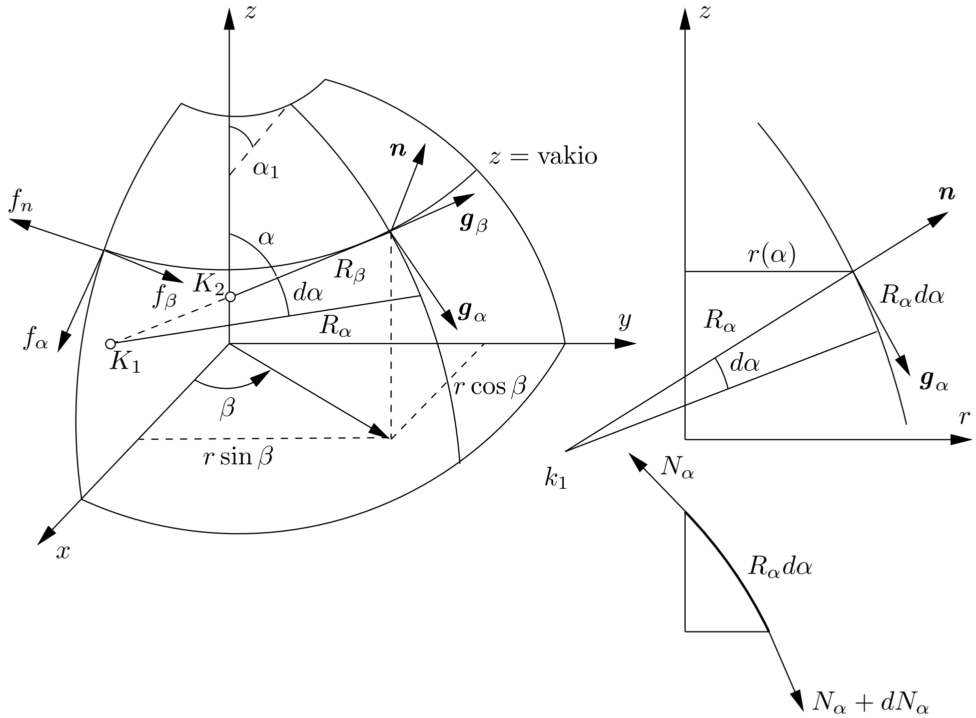
ratkaisu

$$N_\alpha = \frac{A}{r^2}\frac{\partial \Phi}{\partial \beta}, \quad N_\beta = \frac{1}{rA}\frac{\partial^2 r}{\partial z^2}\frac{\partial \Phi}{\partial \beta}, \quad N_{\alpha\beta} = -\frac{\partial}{\partial z}\left(\frac{\Phi}{r}\right). \tag{2.257}$$

Tällöin ensimmäinen ja kolmas tasapainoyhtälö toteutuvat ilman muuta, ja toisesta tasapainoehdosta seuraa

$$\frac{\partial^2 \Phi}{\partial z^2} - \frac{1}{r}\frac{\partial^2 r}{\partial z^2}\left(\Phi + \frac{\partial^2 \Phi}{\partial \beta^2}\right) = 0, \tag{2.258}$$

jonka ratkaisu on homogeenisen pyörähdyskuoren ratkaisu kalvotilassa.



Kuva 2.11 Pyörähdyskuoren geometria parametrin α avulla.

Pyörähdyssymmetrinen kuormitus

$$f_\beta = 0, \quad N_{\alpha\beta} = 0, \tag{2.259}$$

$$\frac{\partial}{\partial z}(rN_\alpha) - \frac{\partial r}{\partial z}N_\beta + r\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}\, f_\alpha = 0, \tag{2.260}$$

$$-\frac{r\dfrac{\partial^2 r}{\partial z^2}}{1+\left(\dfrac{\partial r}{\partial z}\right)^2}N_\alpha + N_\beta - r\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}\, f_n = 0, \tag{2.261}$$

ratkaisut

$$N_\alpha = \frac{1}{r}\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}\left[C + \int_{z_0}^{z} r\left(\frac{\partial r}{\partial z}f_n - f_\alpha\right)dz\right], \tag{2.262}$$

$$N_\beta = \frac{r\dfrac{\partial^2 r}{\partial z^2}}{1+\left(\dfrac{\partial r}{\partial z}\right)^2}N_\alpha + r\sqrt{1+\left(\frac{\partial r}{\partial z}\right)^2}\, f_n. \tag{2.263}$$

2.7.4 Pyörähdyskuoren kalvotila II, $r = r(\alpha)$

$$R_1 = R_\alpha, \quad R_2 = R_\beta = \frac{r}{\sin\alpha}, \tag{2.264}$$

$$ds_\alpha = Ad\alpha = R_\alpha d\alpha, \quad ds_\beta = Bd\beta = rd\beta, \tag{2.265}$$

$$A = R_\alpha, \quad B = r. \tag{2.266}$$

$$dr = R_\alpha d\alpha \cos\alpha \Rightarrow \frac{dr}{d\alpha} = \frac{dB}{d\alpha} = R_\alpha \cos\alpha. \tag{2.267}$$

Tasapainoyhtälöt

$$\frac{\partial}{\partial\alpha}(R_\beta \sin\alpha N_\alpha) - R_\alpha \cos\alpha N_\beta + R_\alpha \frac{\partial N_{\alpha\beta}}{\partial\beta} + R_\alpha R_\beta \sin\alpha f_\alpha = 0, \tag{2.268}$$

$$R_\alpha \frac{\partial N_\beta}{\partial\beta} + \frac{1}{R_\beta \sin\alpha} \frac{\partial}{\partial\alpha}(R_\beta^2 \sin^2\alpha N_{\alpha\beta}) + R_\alpha R_\beta \sin\alpha f_\beta = 0, \tag{2.269}$$

$$\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} - f_n = 0. \tag{2.270}$$

Pyörähdyssymmetrisessä tapauksessa $f_\beta = 0$, $N_{\alpha\beta} = 0$

$$\frac{d}{d\alpha}(R_\beta \sin\alpha N_\alpha) - R_\alpha \cos\alpha N_\beta + R_\alpha R_\beta \sin\alpha f_\alpha = 0, \tag{2.271}$$

$$\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} - f_n = 0, \tag{2.272}$$

$$N_\alpha = \frac{1}{R_\beta \sin^2\alpha}[C + \int_{\alpha_1}^{\alpha_2} R_\alpha R_\beta \sin\alpha (f_n \cos\alpha - f_\alpha \sin\alpha)\, d\alpha], \tag{2.273}$$

$$N_\beta = R_\beta f_n - \frac{R_\beta}{R_\alpha} N_\alpha. \tag{2.274}$$

Leikkauksen $z =$ vakio yläpuolella olevan osan pystysuora tasapainoehto

$$Q_z + 2\pi r N_\alpha \sin\alpha = 0, \tag{2.275}$$

$$N_\alpha = -\frac{Q_z}{2\pi r \sin\alpha}, \quad N_\beta = R_\beta \left(f_n - \frac{N_\alpha}{R_\alpha} \right). \tag{2.276}$$

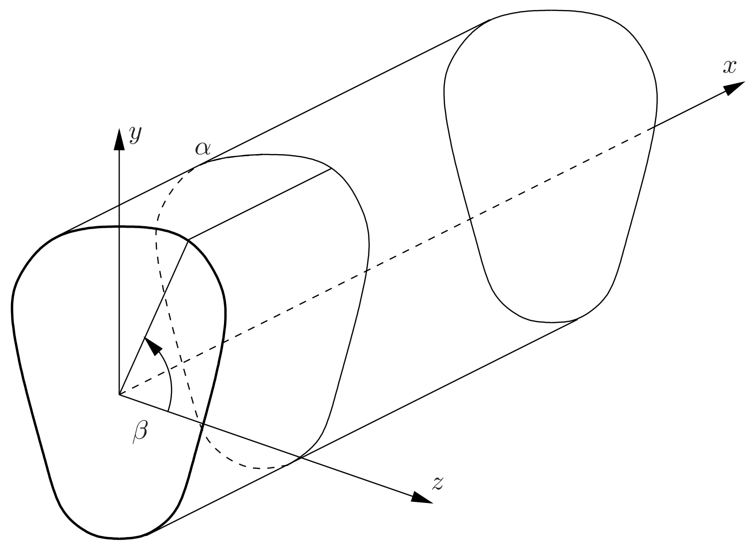
Sylinteri- ja kartiokuori

$$\mathbf{r} = \alpha\mathbf{i} + y(\beta)\mathbf{j} + z(\beta)\mathbf{k}. \tag{2.277}$$

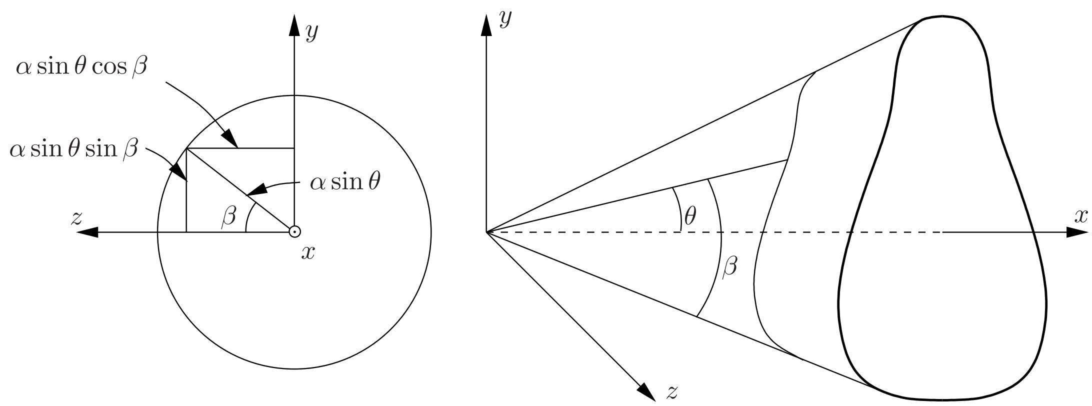
$$A = 1, \quad B = \sqrt{\left(\frac{\partial y}{\partial\beta}\right)^2 + \left(\frac{\partial z}{\partial\beta}\right)^2}, \quad F = 0. \tag{2.278}$$

$$L = 0, \quad M = 0, \quad N = \frac{1}{B}\left(\frac{\partial y}{\partial\beta}\frac{\partial^2 z}{\partial\beta^2} - \frac{\partial z}{\partial\beta}\frac{\partial^2 y}{\partial\beta^2} \right), \tag{2.279}$$

$$k_1 = \frac{1}{R_1} = 0, \quad k_2 = \frac{1}{R_2} = \frac{\dfrac{\partial y}{\partial\beta}\dfrac{\partial^2 z}{\partial\beta^2} - \dfrac{\partial z}{\partial\beta}\dfrac{\partial^2 y}{\partial\beta^2}}{\left[\left(\dfrac{\partial y}{\partial\beta}\right)^2 + \left(\dfrac{\partial z}{\partial\beta}\right)\right]^{\frac{3}{2}}}. \tag{2.280}$$



Kuva 2.12 Sylinterikuori.



Kuva 2.13 Kartiokuori.

Kartiopinta

$$\mathbf{r} = \alpha\cos\theta\mathbf{i} + \alpha\sin\theta\sin\beta\mathbf{j} + \alpha\sin\theta\cos\beta\mathbf{k}, \tag{2.281}$$

$$\theta = \theta(\beta) \tag{2.282}$$

$$\alpha^2 = x^2 + y^2 + z^2, \quad \frac{y}{z} = \tan\beta. \tag{2.283}$$

Kartiokuori

$$A = 1, \quad B = \alpha\sqrt{\sin^2\theta + \left(\frac{\partial\theta}{\partial\beta}\right)^2}, \quad F = 0, \tag{2.284}$$

$$L = 0, \quad M = 0, \quad N = -\frac{\alpha^2}{B}\left[\cos\theta\sin^2\theta + 2\left(\frac{\partial\theta}{\partial\beta}\right)^2\cos\theta - \frac{\partial^2\theta}{\partial\beta^2}\sin\theta\right], \tag{2.285}$$

$$k_1 = \frac{1}{R_1} = 0, \tag{2.286}$$

$$k_2 = \frac{1}{R_2} = -\frac{\cos\theta\sin^2\theta + 2\left(\frac{\partial\theta}{\partial\beta}\right)^2\cos\theta - \frac{\partial^2\theta}{\partial\beta^2}\sin\theta}{\alpha\left[\sin^2\theta + \left(\frac{\partial\theta}{\partial\beta}\right)^2\right]^{\frac{3}{2}}}. \tag{2.287}$$

Tasapainoyhtälöt

$$\frac{\partial}{\partial\alpha}(BN_\alpha) - \frac{\partial B}{\partial\alpha}N_\beta + \frac{\partial N_{\alpha\beta}}{\partial\beta} + Bf_\alpha = 0, \tag{2.288}$$

$$\frac{\partial N_\beta}{\partial\beta} + \frac{1}{B}\frac{\partial}{\partial\alpha}(B^2 N_{\alpha\beta}) + Bf_\beta = 0, \tag{2.289}$$

$$\frac{N_\beta}{R_\beta} - f_n = 0 \Rightarrow N_\beta = R_\beta f_n, \tag{2.290}$$

$$N_{\alpha\beta} = \frac{1}{B^2}f_1(\beta) - \frac{1}{B^2}\int_{\alpha_1}^{\alpha} B\left[\frac{\partial(R_\beta f_n)}{\partial\beta} + Bf_\beta\right]d\alpha, \tag{2.291}$$

$$\begin{aligned} N_\alpha =& -\frac{1}{B}\int_{\alpha_1}^{\alpha}\frac{\partial}{\partial\beta}\left[\frac{f_1(\beta)}{B^2}\right]d\alpha + \frac{f_2(\beta)}{B} + \frac{1}{B}\int_{\alpha_1}^{\alpha}\left(\frac{\partial B}{\partial\alpha}R_\beta f_n - Bf_\alpha\right)d\alpha \\ &+ \frac{1}{B}\int_{\alpha_1}^{\alpha}\frac{\partial}{\partial\beta}\left\{\frac{1}{B^2}\int_{\alpha_1}^{\alpha}B\left[\frac{\partial(R_\beta f_n)}{\partial\beta} + Bf_\beta\right]d\alpha\right\}d\alpha, \end{aligned} \tag{2.292}$$

f_1 ja f_2 ovat integroimisvakioita (riippuvat kuitenkin koordinaatista β), α_1 on vakio.

2.7.5 Pyörähdyskuoren siirtymät kalvotilassa

$$\varepsilon_\alpha = \frac{1}{R_\alpha}\frac{du}{d\alpha} + \frac{w}{R_\alpha}, \quad \varepsilon_\beta = \frac{u\cos\alpha + w\sin\alpha}{r}, \tag{2.293}$$

$$r = R_\beta\sin\alpha \Rightarrow \varepsilon_\beta = \frac{1}{R_\beta}(u\cot\alpha + w). \tag{2.294}$$

$$\Rightarrow \frac{du}{d\alpha} - u\cot\alpha = R_\alpha\varepsilon_\alpha - R_\beta\varepsilon_\beta, \tag{2.295}$$

$$\sin\alpha\frac{d\left(\frac{u}{\sin\alpha}\right)}{d\alpha} = R_\alpha\varepsilon_\alpha - R_\beta\varepsilon_\beta, \tag{2.296}$$

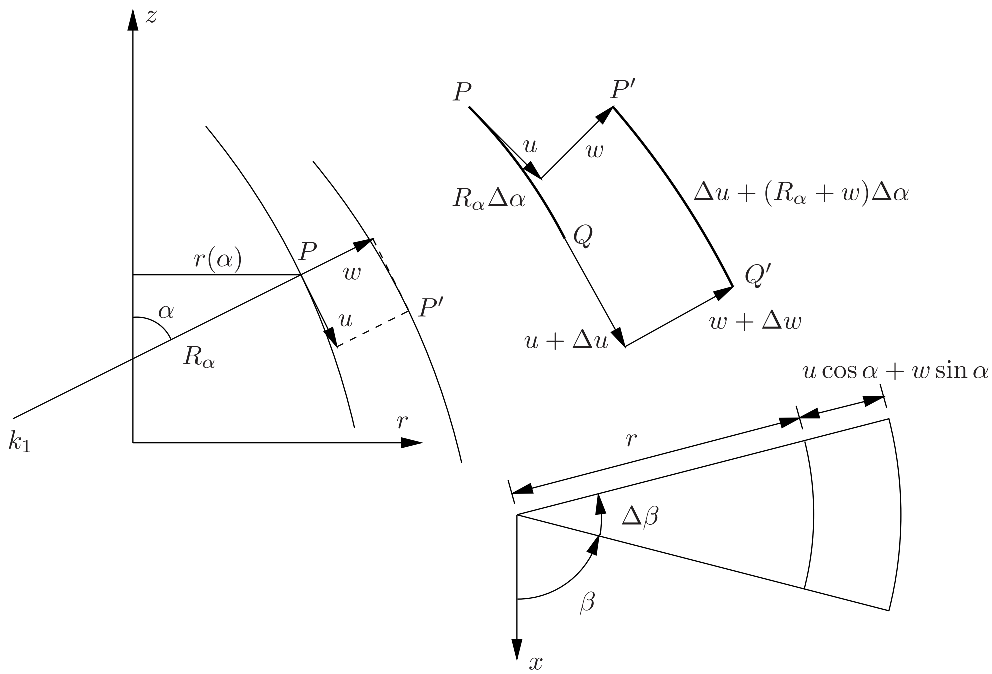
$$\Rightarrow u = \sin\alpha\left[\int(R_\alpha\varepsilon_\alpha - R_\beta\varepsilon_\beta)\frac{1}{\sin\alpha}\,d\alpha + C\right]. \tag{2.297}$$

Normaalin suuntainen siirtymä

$$w = -u\cot\alpha + R_\beta\varepsilon_\beta. \tag{2.298}$$

Kuoren reunan kiertymä

$$\varphi = \frac{1}{R_\alpha}\left(u - \frac{dw}{d\alpha}\right) = -\cot\alpha(\varepsilon_\beta - \varepsilon_\alpha) - \frac{R_\beta}{R_\alpha}\frac{d\varepsilon_\beta}{d\alpha}. \tag{2.299}$$



Kuva 2.14 Pyörähdyskuoren kalvotilan siirtymät.

2.8 Yleinen kuoriteoria

2.8.1 Tasapainoyhtälöt

$$\frac{\partial}{\partial\alpha}(BN_\alpha) - \frac{\partial B}{\partial\alpha}N_\beta + \frac{\partial}{\partial\beta}(AN_{\beta\alpha}) + \frac{\partial A}{\partial\beta}N_{\alpha\beta} + \frac{AB}{R_\alpha}Q_\alpha + ABf_\alpha = 0, \tag{2.300}$$

$$\frac{\partial}{\partial\beta}(AN_\beta) - \frac{\partial A}{\partial\beta}N_\alpha + \frac{\partial}{\partial\alpha}(BN_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}N_{\beta\alpha} + \frac{AB}{R_\beta}Q_\beta + ABf_\beta = 0, \tag{2.301}$$

$$-\frac{AB}{R_\alpha}N_\alpha - \frac{AB}{R_\beta}N_\beta + \frac{\partial}{\partial\alpha}(BQ_\alpha) + \frac{\partial}{\partial\beta}(AQ_\beta) + ABf_n = 0. \tag{2.302}$$

Momenttien tasapainoehdot

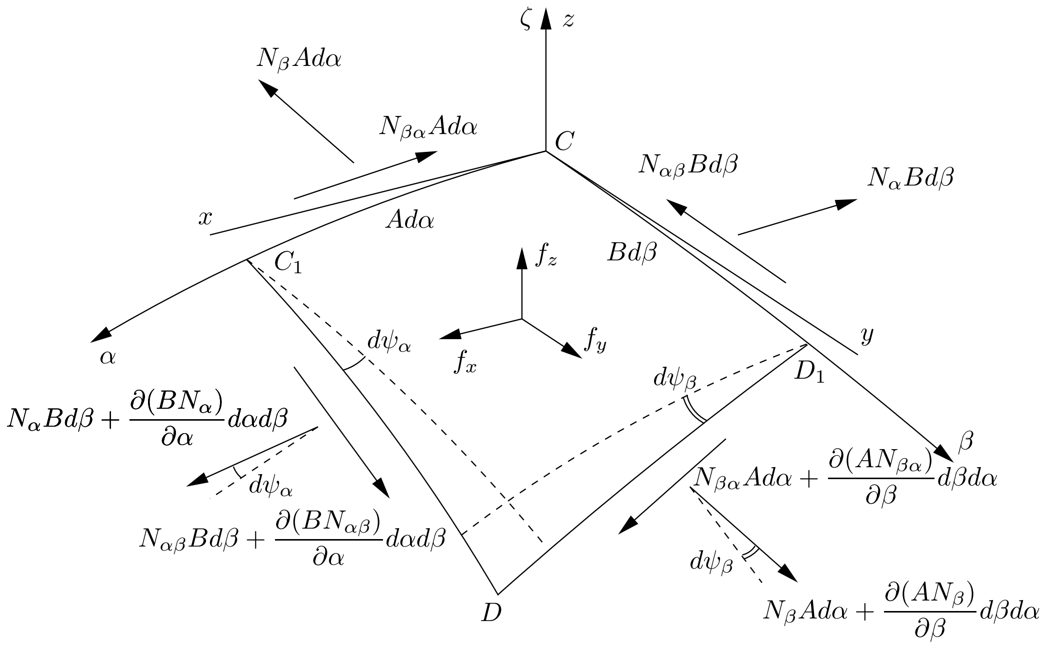
$$\frac{\partial}{\partial\alpha}(BM_\alpha) - \frac{\partial B}{\partial\alpha}M_\beta + \frac{\partial}{\partial\beta}(AM_{\beta\alpha}) + \frac{\partial A}{\partial\beta}M_{\alpha\beta} - ABQ_\alpha = 0, \tag{2.303}$$

$$\frac{\partial}{\partial\beta}(AM_\beta) - \frac{\partial A}{\partial\beta}M_\alpha + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}M_{\beta\alpha} - ABQ_\beta = 0, \tag{2.304}$$

$$N_{\alpha\beta} - N_{\beta\alpha} + \frac{M_{\alpha\beta}}{R_\alpha} - \frac{M_{\beta\alpha}}{R_\beta} = 0. \tag{2.305}$$

Eliminoimalla leikkausvoimat kolme tasapainoyhtälöä

$$\begin{aligned}&\frac{\partial}{\partial\alpha}(BN_\alpha) - \frac{\partial B}{\partial\alpha}N_\beta + \frac{\partial}{\partial\beta}(AN_{\beta\alpha}) + \frac{\partial A}{\partial\beta}N_{\alpha\beta}\\ &+\frac{1}{R_\alpha}\left[\frac{\partial}{\partial\alpha}(BM_\alpha) - \frac{\partial B}{\partial\alpha}M_\beta + \frac{\partial}{\partial\beta}(AM_{\beta\alpha}) + \frac{\partial A}{\partial\beta}M_{\alpha\beta}\right] + ABf_\alpha = 0,\end{aligned} \tag{2.306}$$



Kuva 2.15 Jännitysresultantit.

$$\begin{aligned}&\frac{\partial}{\partial\beta}(AN_\beta) - \frac{\partial A}{\partial\beta}N_\alpha + \frac{\partial}{\partial\alpha}(BN_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}N_{\beta\alpha}\\ &+\frac{1}{R_\beta}\left[\frac{\partial}{\partial\beta}(AM_\beta) - \frac{\partial A}{\partial\beta}M_\alpha + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}M_{\beta\alpha}\right] + ABf_\beta = 0,\end{aligned} \tag{2.307}$$

$$\begin{aligned}&-\frac{N_\alpha}{R_\alpha} - \frac{N_\beta}{R_\beta}\\ &+\frac{1}{AB}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[\frac{\partial}{\partial\alpha}(BM_\alpha) - \frac{\partial B}{\partial\alpha}M_\beta + \frac{\partial}{\partial\beta}(AM_{\beta\alpha}) + \frac{\partial A}{\partial\beta}M_{\alpha\beta}\right]\right\}\\ &+\frac{1}{AB}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[\frac{\partial}{\partial\beta}(AM_\beta) - \frac{\partial A}{\partial\beta}M_\alpha + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}M_{\beta\alpha}\right]\right\} + f_n = 0.\end{aligned} \tag{2.308}$$

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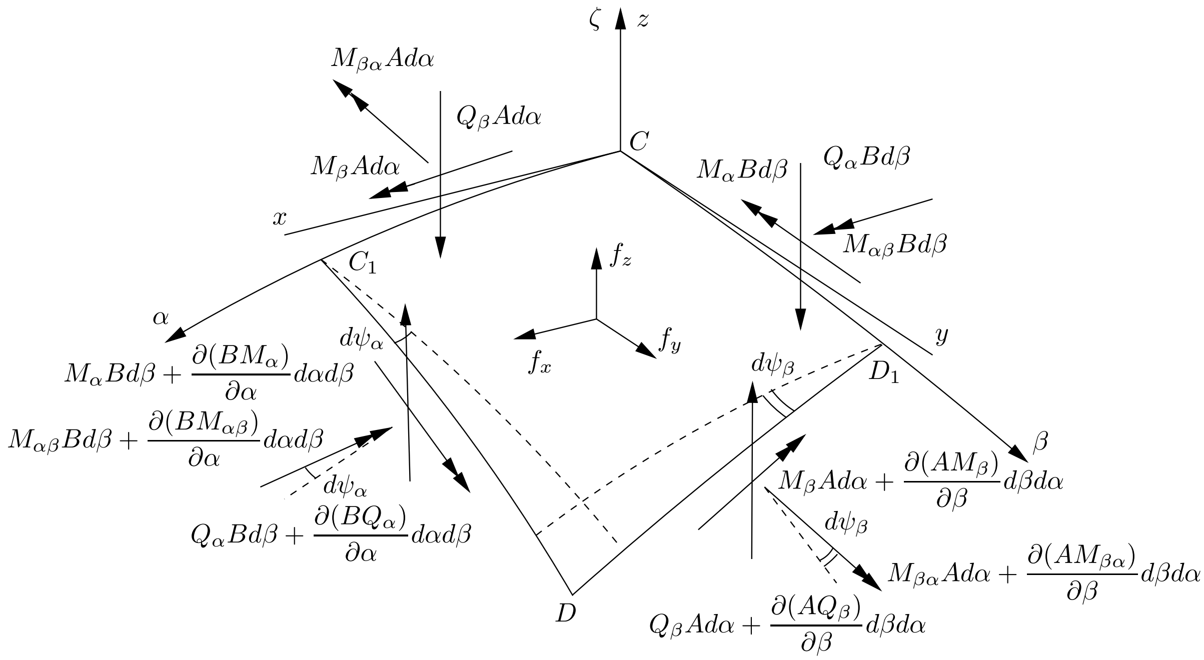
$$N_{\alpha\beta} = N_{\beta\alpha}, \quad M_{\alpha\beta} = M_{\beta\alpha} \Rightarrow \tag{2.309}$$

$$\begin{aligned}&\frac{\partial}{\partial\alpha}(BN_\alpha) - \frac{\partial B}{\partial\alpha}N_\beta + \frac{1}{A}\frac{\partial}{\partial\beta}(A^2N_{\alpha\beta})\\ &+\frac{1}{R_\alpha}\left[\frac{\partial}{\partial\alpha}(BM_\alpha) + \frac{1}{A}\frac{\partial}{\partial\beta}(A^2M_{\alpha\beta}) - \frac{\partial B}{\partial\alpha}M_\beta\right] + ABf_\alpha = 0,\end{aligned} \tag{2.310}$$

$$\begin{aligned}&\frac{\partial}{\partial\beta}(AN_\beta) - \frac{\partial A}{\partial\beta}N_\alpha + \frac{1}{B}\frac{\partial}{\partial\alpha}(B^2N_{\alpha\beta})\\ &+\frac{1}{R_\beta}\left[\frac{\partial}{\partial\beta}(AM_\beta) + \frac{1}{B}\frac{\partial}{\partial\alpha}(B^2M_{\alpha\beta}) - \frac{\partial A}{\partial\beta}M_\alpha\right] + ABf_\beta = 0,\end{aligned} \tag{2.311}$$



Kuva 2.16 Kuorialkion taivutusmomentit, vääntömomentit ja leikkausvoimat.

$$\begin{aligned}
-\frac{N_\alpha}{R_\alpha} - \frac{N_\beta}{R_\beta} &+ \frac{1}{AB}\frac{\partial}{\partial\alpha}\left\{\frac{1}{A}\left[\frac{\partial}{\partial\alpha}(BM_\alpha) + \frac{1}{A}\frac{\partial}{\partial\beta}(A^2M_{\alpha\beta}) - \frac{\partial B}{\partial\alpha}M_\beta\right]\right\} \\
&+ \frac{1}{AB}\frac{\partial}{\partial\beta}\left\{\frac{1}{B}\left[\frac{\partial}{\partial\beta}(AM_\beta) + \frac{1}{B}\frac{\partial}{\partial\alpha}(B^2M_{\alpha\beta}) - \frac{\partial A}{\partial\beta}M_\alpha\right]\right\} + f_n = 0.
\end{aligned} \tag{2.312}$$

2.8.2 Kuoren muodonmuutokset

Yksikkötangenttivektorit

$$\mathbf{e}_\alpha = \frac{1}{A}\frac{\partial\mathbf{r}}{\partial\alpha}, \quad \mathbf{e}_\beta = \frac{1}{B}\frac{\partial\mathbf{r}}{\partial\beta}. \tag{2.313}$$

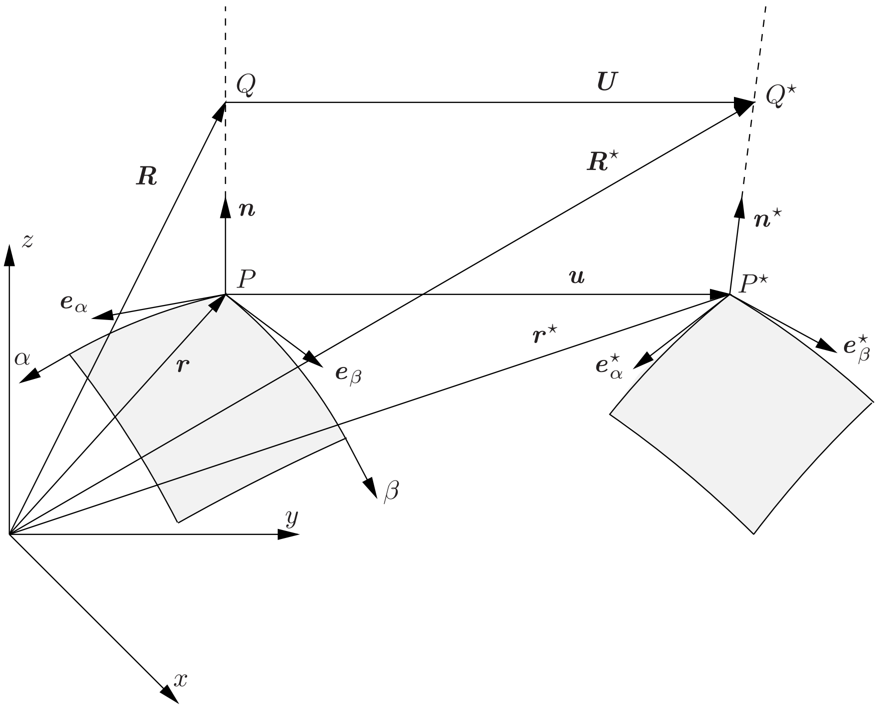
Kaarialkion neliö keskipinnalla

$$ds^2 = A^2(d\alpha)^2 + B^2(d\beta)^2, \tag{2.314}$$

A ja B Lamén parametrit. Keskipinnan yksikkönormaalivektori

$$\mathbf{n} = \mathbf{e}_\alpha \times \mathbf{e}_\beta, \tag{2.315}$$

$$\frac{\partial}{\partial\alpha}\begin{bmatrix}\mathbf{e}_\alpha\\ \mathbf{e}_\beta\\ \mathbf{n}\end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{B}\frac{\partial A}{\partial\beta} & -\frac{A}{R_\alpha} \\ \frac{1}{B}\frac{\partial A}{\partial\beta} & 0 & 0 \\ \frac{A}{R_\alpha} & 0 & 0 \end{bmatrix}\begin{bmatrix}\mathbf{e}_\alpha\\ \mathbf{e}_\beta\\ \mathbf{n}\end{bmatrix}, \tag{2.316}$$



Kuva 2.17 Kuoren siirtymät.

$$\frac{\partial}{\partial\beta}\begin{bmatrix}\mathbf{e}_\alpha\\ \mathbf{e}_\beta\\ \mathbf{n}\end{bmatrix}=\begin{bmatrix}0 & \dfrac{1}{A}\dfrac{\partial B}{\partial\alpha} & 0\\ -\dfrac{1}{A}\dfrac{\partial B}{\partial\alpha} & 0 & -\dfrac{B}{R_\beta}\\ 0 & \dfrac{B}{R_\beta} & 0\end{bmatrix}\begin{bmatrix}\mathbf{e}_\alpha\\ \mathbf{e}_\beta\\ \mathbf{n}\end{bmatrix}, \tag{2.317}$$

$$\frac{\partial^2\mathbf{n}}{\partial\alpha\partial\beta}=\frac{\partial^2\mathbf{n}}{\partial\beta\partial\alpha}\Rightarrow \tag{2.318}$$

Codazzin kaavat

$$\frac{\partial}{\partial\alpha}\left(\frac{B}{R_\alpha}\right)=\frac{1}{R_\alpha}\frac{\partial B}{\partial\alpha},\quad \frac{\partial}{\partial\beta}\left(\frac{A}{R_\beta}\right)=\frac{1}{R_\beta}\frac{\partial A}{\partial\beta}, \tag{2.319}$$

$$\frac{\partial^2\mathbf{e}_\alpha}{\partial\alpha\partial\beta}=\frac{\partial^2\mathbf{e}_\alpha}{\partial\beta\partial\alpha},\quad \frac{\partial^2\mathbf{e}_\beta}{\partial\alpha\partial\beta}=\frac{\partial^2\mathbf{e}_\beta}{\partial\beta\partial\alpha}\Rightarrow \tag{2.320}$$

Gaussin yhtälö

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right)+\frac{AB}{R_\alpha R_\beta}=0. \tag{2.321}$$

$$\mathbf{u}=u\mathbf{e}_\alpha+v\mathbf{e}_\beta+w\mathbf{n}, \tag{2.322}$$

$$\mathbf{U}=U\mathbf{e}_\alpha+V\mathbf{e}_\beta+W\mathbf{n}. \tag{2.323}$$

$$\mathbf{U}=\mathbf{u}+(\mathbf{n}^*-\mathbf{n})\zeta, \tag{2.324}$$

$$\mathbf{n}^*=\mathbf{e}_\alpha^*\times\mathbf{e}_\beta^*, \tag{2.325}$$

$$\mathbf{r}^*=\mathbf{r}+\mathbf{u}=\mathbf{r}+u\mathbf{e}_\alpha+v\mathbf{e}_\beta+w\mathbf{n}, \tag{2.326}$$

$$\mathbf{e}_\alpha^* = \frac{1}{A^*}\frac{\partial \mathbf{r}^*}{\partial \alpha} = \frac{1}{A^*}\left[\frac{\partial \mathbf{r}}{\partial \alpha} + \frac{\partial}{\partial \alpha}(u\mathbf{e}_\alpha + v\mathbf{e}_\beta + w\mathbf{n})\right], \tag{2.327}$$

$$A^* = \sqrt{\frac{\partial \mathbf{r}^*}{\partial \alpha}\cdot\frac{\partial \mathbf{r}^*}{\partial \alpha}}, \tag{2.328}$$

$$\frac{\partial \mathbf{r}^*}{\partial \alpha} \approx A\left(1 + \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_\alpha}\right)\mathbf{e}_\alpha + \left(\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial A}{\partial \beta}u\right)\mathbf{e}_\beta + \left(\frac{\partial w}{\partial \alpha} - \frac{A}{R_\alpha}u\right)\mathbf{n}, \tag{2.329}$$

$$(A^*)^2 = \left|\frac{\partial \mathbf{r}^*}{\partial \alpha}\right|^2 \approx A^2\left(1 + \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_\alpha}\right)^2, \tag{2.330}$$

$$A^* \approx A(1+\varepsilon_\alpha), \quad \varepsilon_\alpha = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_\alpha}, \tag{2.331}$$

$$\Rightarrow \mathbf{e}_\alpha^* = \frac{1}{A^*}\frac{\partial \mathbf{r}^*}{\partial \alpha} \approx \mathbf{e}_\alpha + \left(\frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u\right)\mathbf{e}_\beta + \left(\frac{1}{A}\frac{\partial w}{\partial \alpha} - \frac{u}{R_\alpha}\right)\mathbf{n}. \tag{2.332}$$

Samalla tavalla

$$\mathbf{e}_\beta^* = \frac{1}{B^*}\frac{\partial \mathbf{r}^*}{\partial \beta} = \frac{1}{B^*}\left[\frac{\partial \mathbf{r}}{\partial \beta} + \frac{\partial}{\partial \beta}(u\mathbf{e}_\alpha + v\mathbf{e}_\beta + w\mathbf{n})\right], \tag{2.333}$$

$$B^* = \sqrt{\frac{\partial \mathbf{r}^*}{\partial \beta}\cdot\frac{\partial \mathbf{r}^*}{\partial \beta}} \Rightarrow \tag{2.334}$$

$$\frac{\partial \mathbf{r}^*}{\partial \beta} \approx \left(\frac{\partial u}{\partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}v\right)\mathbf{e}_\alpha + B\left(1 + \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_\beta}\right)\mathbf{e}_\beta + \left(\frac{\partial w}{\partial \beta} - \frac{B}{R_\beta}v\right)\mathbf{n}, \tag{2.335}$$

$$(B^*)^2 = \left|\frac{\partial \mathbf{r}^*}{\partial \beta}\right|^2 \approx B^2\left(1 + \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_\beta}\right)^2, \tag{2.336}$$

$$B^* \approx B(1+\varepsilon_\beta), \quad \varepsilon_\beta = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_\beta}, \tag{2.337}$$

$$\mathbf{e}_\beta^* = \frac{1}{B^*}\frac{\partial \mathbf{r}^*}{\partial \beta} \approx \mathbf{e}_\beta + \left(\frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}v\right)\mathbf{e}_\alpha + \left(\frac{1}{B}\frac{\partial w}{\partial \beta} - \frac{v}{R_\beta}\right)\mathbf{n} \tag{2.338}$$

$$\Rightarrow \mathbf{n}^* = \mathbf{e}_\alpha^* \times \mathbf{e}_\beta^* \approx \mathbf{n} + \varphi\mathbf{e}_\alpha + \psi\mathbf{e}_\beta, \tag{2.339}$$

$$\varphi = -\frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_\alpha}, \tag{2.340}$$

$$\psi = -\frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_\beta} \tag{2.341}$$

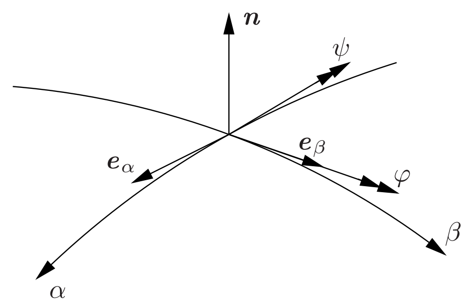
normaalivektorin $\mathbf{n}$ kiertymät tangenttivektoreiden $\mathbf{e}_\beta$ ja $\mathbf{e}_\alpha$ ympäri

$$\Rightarrow \mathbf{U} = \mathbf{u} + \zeta(\varphi\mathbf{e}_\alpha + \psi\mathbf{e}_\beta), \tag{2.342}$$

$$U = u + \zeta\varphi, \quad V = v + \zeta\psi, \quad W = w. \tag{2.343}$$

Viiva–alkion ds_α venymä

$$\varepsilon_\alpha = \frac{ds_\alpha^* - ds_\alpha}{ds_\alpha} = \frac{A^* d\alpha - A d\alpha}{A d\alpha}, \tag{2.344}$$



Kuva 2.18 Kuoren keskipinnan kiertymät.

$$\varepsilon_\alpha = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_\alpha}, \tag{2.345}$$

$$\varepsilon_\beta = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_\beta}. \tag{2.346}$$

Keskipinnan leikkausmuodonmuutos

$$\omega \approx \mathbf{e}_\alpha^* \cdot \mathbf{e}_\beta^*, \tag{2.347}$$

$$\omega = \frac{B}{A}\frac{\partial}{\partial \alpha}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial \beta}\left(\frac{u}{A}\right). \tag{2.348}$$

Etäisyydellä ζ kuoren keskipinnalta

$$R_\alpha^\zeta = R_\alpha + \zeta, \quad R_\beta^\zeta = R_\beta + \zeta, \tag{2.349}$$

$$ds_\alpha^\zeta = A\left(1 + \frac{\zeta}{R_\alpha}\right) = A^\zeta d\alpha, \tag{2.350}$$

$$ds_\beta^\zeta = B\left(1 + \frac{\zeta}{R_\beta}\right) = B^\zeta d\beta, \tag{2.351}$$

$$\varepsilon_\alpha^\zeta = \frac{1}{A\left(1+\frac{\zeta}{R_\alpha}\right)}\frac{\partial U}{\partial \alpha} + \frac{1}{AB\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)}\frac{\partial\left[A\left(1+\frac{\zeta}{R_\alpha}\right)\right]}{\partial \beta}V + \frac{W}{R_\alpha + \zeta}, \tag{2.352}$$

$$\frac{\partial}{\partial \beta}\left(\frac{A}{R_\beta}\right) = \frac{1}{R_\beta}\frac{\partial A}{\partial \beta} \Rightarrow \tag{2.353}$$

$$\frac{\partial\left[A\left(1+\frac{\zeta}{R_\alpha}\right)\right]}{\partial \beta} = \frac{\partial A}{\partial \beta} + \zeta\frac{\partial}{\partial \beta}\left(\frac{A}{R_\alpha}\right) = \left(1+\frac{\zeta}{R_\beta}\right)\frac{\partial A}{\partial \beta}, \tag{2.354}$$

$$\Rightarrow \varepsilon_\alpha^\zeta = \frac{1}{1+\frac{\zeta}{R_\alpha}}\left(\frac{1}{A}\frac{\partial U}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}V + \frac{W}{R_\alpha}\right), \tag{2.355}$$

$$\varepsilon_\alpha^\zeta = \frac{1}{1+\frac{\zeta}{R_\alpha}}\left(\varepsilon_\alpha + \zeta\kappa_\alpha\right), \quad \varepsilon_\beta^\zeta = \frac{1}{1+\frac{\zeta}{R_\beta}}\left(\varepsilon_\beta + \zeta\kappa_\beta\right), \tag{2.356}$$

$$\kappa_\alpha = \frac{1}{A}\frac{\partial \varphi}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\psi, \quad \kappa_\beta = \frac{1}{B}\frac{\partial \psi}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\varphi. \tag{2.357}$$

Leikkausmuodonmuutos keskipinnan suuntaisella pinnalla

$$\omega^\zeta = \frac{1}{\left(1+\frac{\zeta}{R_\alpha}\right)\left(1+\frac{\zeta}{R_\beta}\right)}\left\{\left(1+\frac{\zeta^2}{R_\alpha R_\beta}\right)\omega + 2\left[1+\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)\frac{\zeta}{2}\right]\zeta\kappa_{\alpha\beta}\right\}, \tag{2.358}$$

$$\kappa_{\alpha\beta} = \frac{1}{2}\left\{\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{\psi}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{\varphi}{A}\right)\right\}. \tag{2.359}$$

Ohuen kuoren tapauksessa

$$\frac{\zeta}{R_\alpha} \ll 1, \quad \frac{\zeta}{R_\beta} \ll 1, \tag{2.360}$$

$$\begin{bmatrix} \varepsilon_\alpha^\zeta \\ \varepsilon_\beta^\zeta \\ \omega^\zeta \end{bmatrix} = \begin{bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \omega \end{bmatrix} + \zeta \begin{bmatrix} \kappa_\alpha \\ \kappa_\beta \\ 2\kappa_{\alpha\beta} \end{bmatrix}, \tag{2.361}$$

$$\varepsilon_\alpha = \frac{1}{A}\frac{\partial u}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}v + \frac{w}{R_\alpha}, \tag{2.362}$$

$$\varepsilon_\beta = \frac{1}{B}\frac{\partial v}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}u + \frac{w}{R_\beta}, \tag{2.363}$$

$$\omega = \frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right), \tag{2.364}$$

$$\kappa_\alpha = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(-\frac{1}{A}\frac{\partial w}{\partial\alpha} + \frac{u}{R_\alpha}\right) + \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(-\frac{1}{B}\frac{\partial w}{\partial\beta} + \frac{v}{R_\beta}\right), \tag{2.365}$$

$$\kappa_\beta = \frac{1}{B}\frac{\partial}{\partial\beta}\left(-\frac{1}{B}\frac{\partial w}{\partial\beta} + \frac{v}{R_\beta}\right) + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(-\frac{1}{A}\frac{\partial w}{\partial\alpha} + \frac{u}{R_\alpha}\right), \tag{2.366}$$

$$2\kappa_{\alpha\beta} = \frac{B}{A}\frac{\partial}{\partial\alpha}\left(-\frac{1}{B^2}\frac{\partial w}{\partial\beta} + \frac{v}{BR_\beta}\right) + \frac{A}{B}\frac{\partial}{\partial\beta}\left(-\frac{1}{A^2}\frac{\partial w}{\partial\alpha} + \frac{u}{AR_\alpha}\right). \tag{2.367}$$

2.8.3 Pyörähdyskuori

$$r = r(\alpha), \quad R_1 = R_\alpha, \quad R_2 = R_\beta = \frac{r}{\sin\alpha}. \tag{2.368}$$

$$ds_\alpha = Ad\alpha = R_\alpha d\alpha, \quad ds_\beta = Bd\beta = rd\beta, \tag{2.369}$$

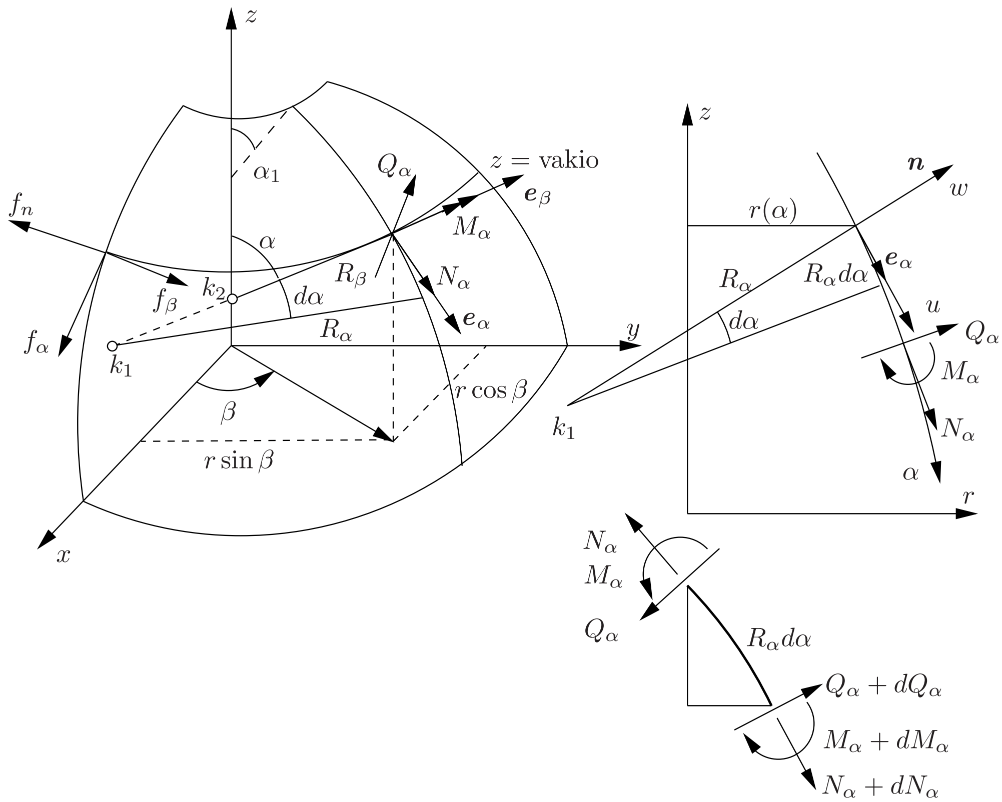
$$A = R_\alpha, \quad B = r = R_\beta \sin\alpha, \tag{2.370}$$

$$dr = R_\alpha d\alpha \cos\alpha \Rightarrow \frac{dr}{d\alpha} = \frac{dB}{d\alpha} = R_\alpha \cos\alpha. \tag{2.371}$$

Pyörähdyssymmetrinen kuormitus

$$f_\beta = 0, \quad N_{\alpha\beta} = Q_\beta = M_{\alpha\beta} = 0, \tag{2.372}$$

$$v = \varepsilon_{\alpha\beta} = \kappa_{\alpha\beta} = 0. \tag{2.373}$$



Kuva 2.19 Pyörähdyskuori.

Tasapainoyhtälöt

$$\frac{d}{d\alpha}(R_\beta \sin\alpha N_\alpha) - R_\alpha \cos\alpha N_\beta + R_\beta \sin\alpha Q_\alpha + R_\alpha R_\beta \sin\alpha f_\alpha = 0, \tag{2.374}$$

$$\frac{d}{d\alpha}(R_\beta \sin\alpha Q_\alpha) - R_\beta \sin\alpha N_\alpha - R_\alpha \sin\alpha N_\beta + R_\alpha R_\beta \sin\alpha f_n = 0, \tag{2.375}$$

$$\frac{d}{d\alpha}(R_\beta \sin\alpha M_\alpha) - R_\alpha \cos\alpha M_\beta - R_\alpha R_\beta \sin\alpha Q_\alpha = 0. \tag{2.376}$$

Muodonmuutosten kaavat

$$\varepsilon_\alpha = \frac{1}{R_\alpha}\frac{du}{d\alpha} + \frac{w}{R_\alpha}, \tag{2.377}$$

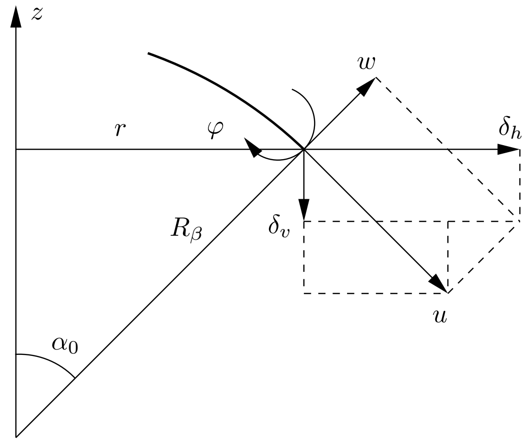
$$\varepsilon_\beta = \frac{1}{R_\beta}(u\cot\alpha + w) = \frac{u\cos\alpha + w\sin\alpha}{r}, \tag{2.378}$$

$$\kappa_\alpha = \frac{1}{R_\alpha}\frac{d\varphi}{d\alpha} = \frac{1}{R_\alpha}\frac{d}{d\alpha}\left[\frac{1}{R_\alpha}\left(u - \frac{dw}{d\alpha}\right)\right], \tag{2.379}$$

$$\kappa_\beta = \frac{1}{R_\alpha R_\beta \sin\alpha}\frac{d(R_\beta \sin\alpha)}{d\alpha}\left[\frac{1}{R_\alpha}\left(u - \frac{dw}{d\alpha}\right)\right] = \frac{\cot\alpha}{R_\alpha R_\beta}\left(u - \frac{dw}{d\alpha}\right), \tag{2.380}$$

kiertymä reunalla $\alpha =$ vakio

$$\varphi = \frac{1}{R_\alpha}\left(u - \frac{dw}{d\alpha}\right). \tag{2.381}$$



Kuva 2.20 Pyörähdyskuoren reunan siirtymät ja kiertymä.

Eliminoimalla siirtymä w

$$\frac{du}{d\alpha} - u\cot\alpha = R_\alpha\varepsilon_\alpha - R_\beta\varepsilon_\beta, \tag{2.382}$$

$$\sin\alpha\frac{d\left(\frac{u}{\sin\alpha}\right)}{d\alpha} = R_\alpha\varepsilon_\alpha - R_\beta\varepsilon_\beta, \tag{2.383}$$

$$\Rightarrow u = \sin\alpha\left[\int(R_\alpha\varepsilon_\alpha - R_\beta\varepsilon_\beta)\frac{1}{\sin\alpha}\,d\alpha + C\right]. \tag{2.384}$$

Normaalin suntainen siirtymä

$$w = -u\cot\alpha + R_\beta\varepsilon_\beta. \tag{2.385}$$

Kuoren reunan kiertymä

$$\varphi = \frac{1}{R_\alpha}\left(u - \frac{dw}{d\alpha}\right) = -\cot\alpha(\varepsilon_\beta - \varepsilon_\alpha) - \frac{R_\beta}{R_\alpha}\frac{d\varepsilon_\beta}{d\alpha}. \tag{2.386}$$

Reunan ($\alpha = \alpha_0$) pysty– ja vaakasiirtymät δ_V ja δ_H

$$\delta_V = u\sin\alpha - w\cos\alpha = \frac{u}{\sin\alpha} - R_\beta\varepsilon_\beta\cos\alpha, \tag{2.387}$$

$$\delta_H = u\cos\alpha + w\sin\alpha = R_\beta\varepsilon_\beta\sin\alpha = r\varepsilon_\beta. \tag{2.388}$$

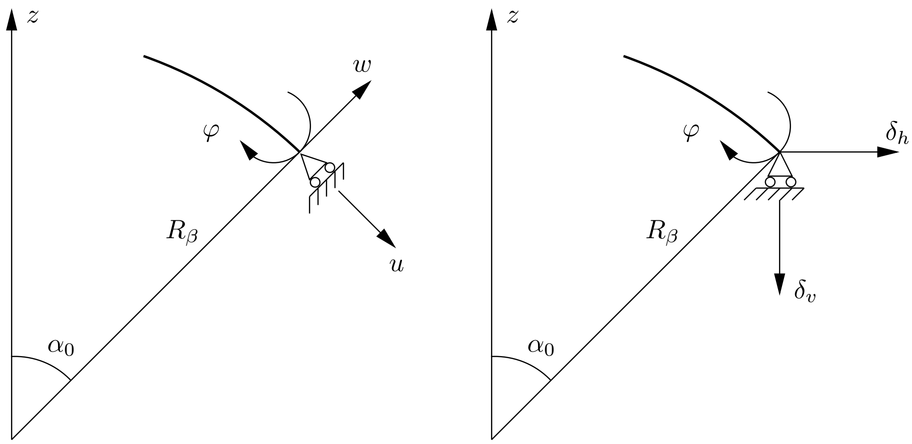
2.8.4 Reunaehdot

1. Siirtymä $u = 0$ reunalla $\alpha = \alpha_0$,

$$w = R_\beta\varepsilon_\beta, \quad \varphi = -\cot\alpha_0(\varepsilon_\beta - \varepsilon_\alpha) - \frac{R_\beta}{R_\alpha}\frac{d\varepsilon_\beta}{d\alpha}. \tag{2.389}$$

2. Siirtymä $\delta_V = 0$ on reunalla,

$$\delta_H = r\varepsilon_\beta, \quad \varphi = -\cot\alpha_0(\varepsilon_\beta - \varepsilon_\alpha) - \frac{R_\beta}{R_\alpha}\frac{d\varepsilon_\beta}{d\alpha}. \tag{2.390}$$



Kuva 2.21 Pyörähdyskuoren reunan siirtymien reunaehdot.

Hooken lain mukaan

$$N_\alpha = \frac{Eh}{1-\nu^2}(\varepsilon_\alpha + \nu\varepsilon_\beta), \quad N_\beta = \frac{Eh}{1-\nu^2}(\varepsilon_\beta + \nu\varepsilon_\alpha), \tag{2.391}$$

$$M_\alpha = \frac{Eh^3}{12(1-\nu^2)}(\kappa_\alpha + \nu\kappa_\beta), \quad M_\beta = \frac{Eh^3}{12(1-\nu^2)}(\kappa_\beta + \nu\kappa_\alpha), \tag{2.392}$$

$$C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)} \tag{2.393}$$

Yleistetystä Hooken laista kääntämällä

$$\varepsilon_\alpha = \frac{1}{Eh}(N_\alpha - \nu N_\beta), \quad \varepsilon_\beta = \frac{1}{Eh}(N_\beta - \nu N_\alpha), \tag{2.394}$$

kiertymä

$$\varphi = -\cot\alpha_0 \frac{1+\nu}{Eh}(N_\beta - N_\alpha) - \frac{R_\beta}{R_\alpha}\frac{1}{Eh}\frac{d}{d\alpha}(N_\beta - \nu N_\alpha). \tag{2.395}$$

2.9 Pyörähdyskuoren reunahäiriö

$$r = r(\alpha), \quad R_1 = R_\alpha, \quad R_2 = R_\beta = \frac{r}{\sin\alpha}. \tag{2.396}$$

$$ds_\alpha = Ad\alpha = R_\alpha d\alpha, \quad ds_\beta = Bd\beta = rd\beta, \tag{2.397}$$

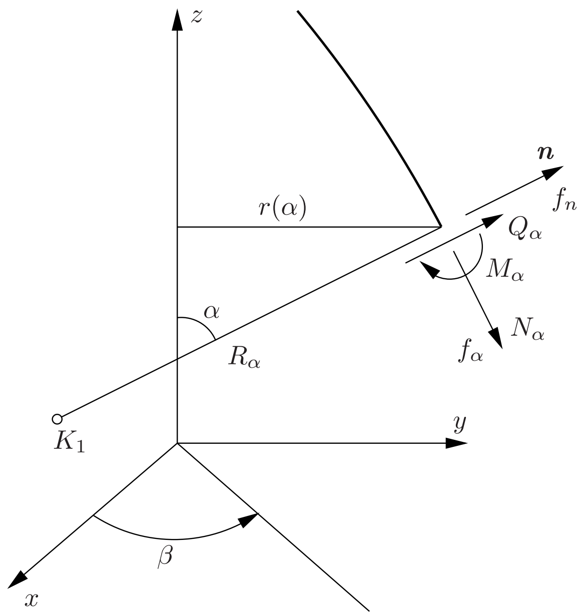
$$A = R_\alpha, \quad B = r = R_\beta \sin\alpha, \tag{2.398}$$

$$dr = R_\alpha d\alpha \cos\alpha \Rightarrow \frac{dr}{d\alpha} = \frac{dB}{d\alpha} = R_\alpha \cos\alpha. \tag{2.399}$$

Pyörähdyssymmetrinen kuormitus

$$f_\beta = 0, \quad N_{\alpha\beta} = Q_\beta = M_{\alpha\beta} = 0, \tag{2.400}$$

$$v = \varepsilon_{\alpha\beta} = \kappa_{\alpha\beta} = 0. \tag{2.401}$$



Kuva 2.22 Pyörähdyskuori.

Tasapainoyhtälöt

$$\frac{d}{d\alpha}(R_\beta \sin\alpha N_\alpha) - R_\alpha \cos\alpha N_\beta + R_\alpha \sin\alpha Q_\alpha + R_\alpha R_\beta \sin\alpha f_\alpha = 0, \tag{2.402}$$

$$\frac{d}{d\alpha}(R_\beta \sin\alpha Q_\alpha) - R_\beta \sin\alpha N_\alpha - R_\alpha \sin\alpha N_\beta + R_\alpha R_\beta \sin\alpha f_n = 0, \tag{2.403}$$

$$\frac{d}{d\alpha}(R_\beta \sin\alpha M_\alpha) - R_\alpha \cos\alpha M_\beta - R_\alpha R_\beta \sin\alpha Q_\alpha = 0 \tag{2.404}$$

tai

$$\frac{d}{d\alpha}(rN_\alpha) - R_\alpha \cos\alpha N_\beta + rQ_\alpha + R_\alpha r f_\alpha = 0, \tag{2.405}$$

$$\frac{d}{d\alpha}(rQ_\alpha) - rN_\alpha - R_\alpha \sin\alpha N_\beta + R_\alpha r f_n = 0, \tag{2.406}$$

$$\frac{d}{d\alpha}(rM_\alpha) - R_\alpha \cos\alpha M_\beta - R_\alpha r Q_\alpha = 0. \tag{2.407}$$

1. Määritetään kalvotilan ratkaisu kuormista f_n ja f_α eli

$$N_\alpha^K, \quad N_\beta^K. \tag{2.408}$$

2. Ratkaistaan homogeeniset tasapainoyhtälöt ja saadaan taivutustilan ratkaisu

$$N_\alpha^T, \quad N_\beta^T, \quad Q_\alpha, \quad M_\alpha, \quad M_\beta, \tag{2.409}$$

Lopullinen ratkaisu on osien 1 ja 2 summa. Eräissä tapauksissa osan 2 ratkaisu reunahäiriötehtävänä. Oletukset:

1. Koordinaatin α suuntainen siirtymä on pieni taipuman rinnalla eli

$$|u| \ll |w|. \tag{2.410}$$

2. Suureet muuttuvat jyrkästi koordinaatin α suunnassa.

Tasapainoyhtälöistä

$$N_\alpha = \cot\alpha Q_\alpha, \quad N_\beta \approx \frac{R_\beta}{R_\alpha}\frac{dQ_\alpha}{d\alpha}, \tag{2.411}$$

$$Q_\alpha \approx \frac{1}{R_\alpha}\frac{dM_\alpha}{d\alpha}, \quad N_\beta \approx \frac{R_\beta}{R_\alpha^2}\frac{d^2M_\alpha}{d\alpha^2}, \tag{2.412}$$

$$\varepsilon_\beta \approx \frac{w}{R_\beta}, \quad \varphi \approx -\frac{1}{R_\alpha}\frac{dw}{d\alpha}, \tag{2.413}$$

$$\kappa_\alpha \approx -\frac{1}{R_\alpha^2}\frac{d^2w}{d\alpha^2}, \quad \kappa_\beta \approx -\frac{\cot\alpha}{R_\alpha R_\beta}\frac{dw}{d\alpha}. \tag{2.414}$$

$$N_\beta \gg N_\alpha \Rightarrow \varepsilon_\beta \approx \frac{1}{Eh}N_\beta \Rightarrow N_\beta \approx Eh\varepsilon_\beta \approx \frac{Eh}{R_\beta}w. \tag{2.415}$$

Momentit

$$M_\alpha \approx \frac{D}{R_\alpha}\frac{d\varphi}{d\alpha}, \quad M_\beta \approx \nu M_\alpha, \tag{2.416}$$

$$N_\beta \approx \frac{R_\beta}{R_\alpha^2}\frac{d^2M_\alpha}{d\alpha^2} \Rightarrow \frac{R_\beta}{R_\alpha^2}\frac{d^2}{d\alpha^2}\left(\frac{D}{R_\alpha^2}\frac{d^2w}{d\alpha^2}\right) + \frac{Eh}{R_\beta}w = 0 \tag{2.417}$$

$$\Rightarrow \frac{R_\beta D}{R_\alpha^4}\frac{d^4w}{d\alpha^4} + \frac{Eh}{R_\beta}w = 0 \Rightarrow \left(\frac{R_\beta}{R_\alpha}\right)^4\frac{d^4w}{d\alpha^4} + 4K^4w = 0, \tag{2.418}$$

$$K = \sqrt[4]{\frac{3(1-\nu^2)R_\beta^2}{h^2}} = \sqrt[4]{3(1-\nu^2)}\sqrt{\frac{R_\beta}{h}} \tag{2.419}$$

Kulman α sijasta muuttujaksi kaarenpituus s: $ds = R_\alpha d\alpha$

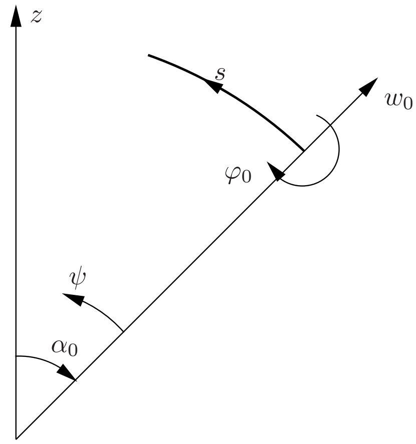
$$\frac{d^4w}{ds^4} + 4\left(\frac{K}{R_\beta}\right)^4 w = 0. \tag{2.420}$$

Reunahäiriödifferentiaaliyhtälön ratkaisu

$$w = e^{-\lambda s}(C_1\cos\lambda s + C_2\sin\lambda s) + e^{\lambda s}(C_3\cos\lambda s + C_4\sin\lambda s), \quad \lambda = \frac{K}{R_\beta}, \tag{2.421}$$

$$\begin{aligned}\frac{dw}{ds} =& \lambda\{e^{-\lambda s}[(-C_1 + C_2)\cos\lambda s + (-C_2 - C_1)\sin\lambda s)] \\ &+ e^{\lambda s}[(C_3 + C_4)\cos\lambda s + (C_4 - C_3)\sin\lambda s]\},\end{aligned} \tag{2.422}$$

$$\begin{aligned}\frac{d^2w}{ds^2} =& 2\lambda^2\{e^{-\lambda s}[-C_2\cos\lambda s + C_1\sin\lambda s] \\ &+ e^{\lambda s}[C_4\cos\lambda s - C_3\sin\lambda s]\},\end{aligned} \tag{2.423}$$



Kuva 2.23 Koordinaatti ψ ja reunan kiertymä ja siirtymä.

$$\frac{d^3w}{ds^3} = 2\lambda^3\{e^{-\lambda s}[(C_1+C_2)\cos\lambda s + (-C_1+C_2)\sin\lambda s)] + e^{\lambda s}[(C_4-C_3)\cos\lambda s + (-C_3-C_4)\sin\lambda s]\}. \tag{2.424}$$

Reunahäiriösuuret

$$N_\beta = \frac{Eh}{R_\beta}w, \quad \varphi = \mp\frac{dw}{ds}, \tag{2.425}$$

$$M_\alpha = -D\frac{d^2w}{ds^2}, \quad Q_\alpha = \mp D\frac{d^3w}{ds^3}, \tag{2.426}$$

valitaan + merkki, jos koordinaattien s ja α suunnat ovat samat ja − merkki, jos koordinaattien s ja α suunnat ovat vastakkaiset. Lausumalla taipuma voiman N_β avulla eli $w = \dfrac{R_\beta}{Eh}N_\beta$ ja ottamalla huomioon, että

$$M_\alpha \approx -\frac{DR_\beta}{EhR_\alpha^2}\frac{d^2N_\beta}{d\alpha^2} \approx -\frac{DR_\beta^2}{EhR_\alpha^3}\frac{d^3Q_\alpha}{d\alpha^3}, \tag{2.427}$$

$$Q_\alpha \approx \frac{1}{R_\alpha}\frac{dM_\alpha}{d\alpha} \Rightarrow \frac{1}{R_\alpha^4}\frac{d^4Q_\alpha}{d\alpha^4} + 4\left(\frac{K}{R_\beta}\right)^4 Q_\alpha = 0 \tag{2.428}$$

$$\Rightarrow \frac{d^4Q_\alpha}{ds^4} + 4\lambda^4 Q_\alpha = 0, \quad \lambda = \frac{K}{R_\beta}. \tag{2.429}$$

Reunahäiriö alueessa $\alpha \le \alpha_0$, uusi muuttuja $\psi = \alpha_0 - \alpha$:

$$\Rightarrow \frac{d^4w}{d\psi^4} + 4K^4\left(\frac{R_\alpha}{R_\beta}\right)^4 w = 0 \Rightarrow \frac{d^4w}{d\psi^4} + 4\gamma^4 w = 0, \tag{2.430}$$

$$w = e^{-\gamma\psi}(C_1\cos\gamma\psi + C_2\sin\gamma\psi) + e^{\gamma\psi}(C_3\cos\gamma\psi + C_4\sin\gamma\psi), \tag{2.431}$$

$$\gamma = \frac{KR_\alpha}{R_\beta}. \tag{2.432}$$

Häiriöt riittävän kaukana toisistaan:

$$w = e^{-\gamma\psi}(C_1\cos\gamma\psi + C_2\sin\gamma\psi), \tag{2.433}$$

$$\frac{dw}{d\alpha} = -\frac{dw}{d\psi} = \gamma e^{-\gamma\psi}[(C_1 - C_2)\cos\gamma\psi + (C_1 + C_2)\sin\gamma\psi)], \tag{2.434}$$

$$\frac{d^2w}{d\alpha^2} = \frac{d^2w}{d\psi^2} = 2\gamma^2 e^{-\gamma\psi}(-C_2\cos\gamma\psi + C_1\sin\gamma\psi), \tag{2.435}$$

$$\frac{d^3w}{d\alpha^3} = -\frac{d^3w}{d\psi^3} = 2\gamma^3 e^{-\gamma\psi}[-(C_1 + C_2)\cos\gamma\psi + (C_1 - C_2)\sin\gamma\psi)]. \tag{2.436}$$

Taivutusmomentti ja leikkausvoima

$$M_\alpha = -D\frac{1}{A^2}\frac{d^2w}{d\psi^2}, \quad Q_\alpha = -\frac{D}{A^3}\frac{d^3w}{d\alpha^3} = \frac{D}{A^3}\frac{d^3w}{d\psi^3}. \tag{2.437}$$

Reunaehdot reunalla $\psi = 0$:

1. Taipuma ja kiertymä on annettu eli

$$w = w_0, \quad \varphi = \varphi_0, \tag{2.438}$$

$$w_0 = w(0) = C_1, \tag{2.439}$$

$$\begin{aligned}\varphi_0 =& \varphi(0) = \frac{1}{A}\frac{dw}{d\psi} + \frac{u}{R_\alpha} \approx \frac{1}{A}\frac{dw}{d\psi} = \frac{1}{R_\alpha}\frac{dw}{d\psi} \\ =& -\frac{\gamma}{R_\alpha}(C_1 - C_2) = -\frac{K}{R_\beta}(C_1 - C_2),\end{aligned} \tag{2.440}$$

$$C_1 = w_0, \quad C_2 = \frac{R_\beta}{K}\varphi_0 + w_0. \tag{2.441}$$

2. Reunalla $\alpha = \alpha_0$ on annettu momentti ja leikkausvoima

$$M_\alpha = M_0, \quad Q_\alpha = Q_0, \tag{2.442}$$

$$M_0 = \frac{2DK^2}{R_\beta^2}C_2, \quad Q_0 = \frac{2DK^3}{R_\beta^3}(C_1 + C_2), \tag{2.443}$$

$$C_1 = -\frac{R_\beta^2}{2DK^2}M_0 + \frac{R_\beta^3}{2DK^3}Q_0, \quad C_2 = \frac{R_\beta^2}{2DK^2}M_0. \tag{2.444}$$

Vaikutuskertoimet tai joustokertoimet

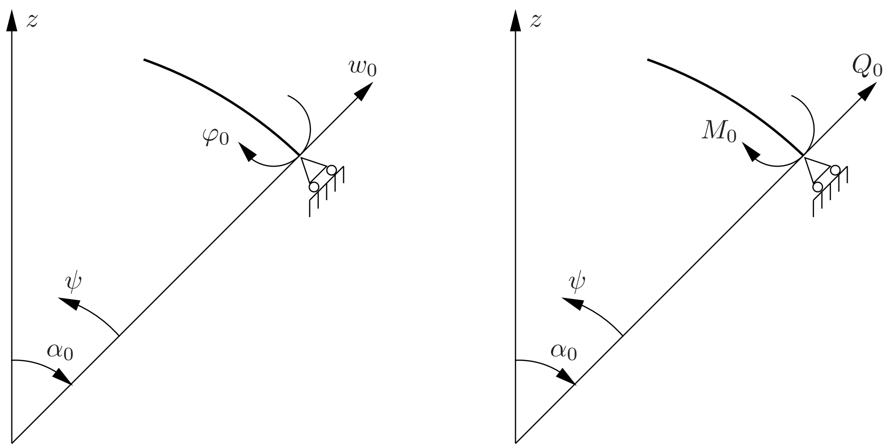
$$\gamma_{11} = \frac{1}{2D}\left(\frac{R_\beta}{K}\right)^3 = \frac{2R_\beta K}{Eh}, \tag{2.445}$$

$$\gamma_{12} = \frac{1}{2D}\left(\frac{R_\beta}{K}\right)^2 = \gamma_{21} = \frac{2K^2}{Eh}, \tag{2.446}$$

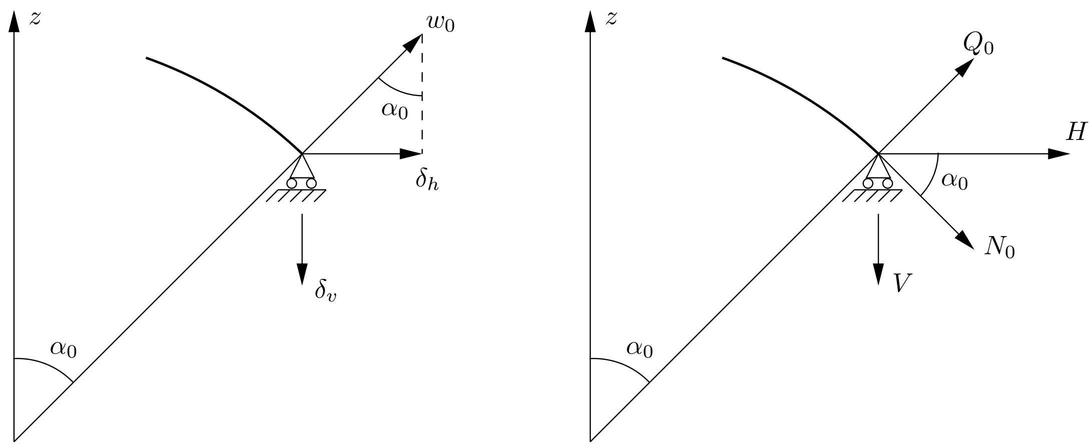
$$\gamma_{22} = \frac{1}{D}\left(\frac{R_\beta}{K}\right) = \frac{4K^3}{EhR_\beta}, \tag{2.447}$$

$$w_0 = \gamma_{11}Q_0 - \gamma_{12}M_0, \tag{2.448}$$

$$\varphi_0 = -\gamma_{12}Q_0 + \gamma_{22}M_0. \tag{2.449}$$



Kuva 2.24 Reunan voima– ja siirtymäsuureet.



Kuva 2.25 Reunan vaaka– ja pystysiirtymä.

Jäykkyyskertoimet:

$$\Gamma_{11} = 4D\left(\frac{K}{R_\beta}\right)^3, \quad \Gamma_{12} = 2D\left(\frac{K}{R_\beta}\right)^2 = \Gamma_{21}, \quad \Gamma_{22} = 2D\left(\frac{K}{R_\beta}\right), \tag{2.450}$$

$$Q_0 = \Gamma_{11}w_0 + \Gamma_{12}\varphi_0, \tag{2.451}$$

$$M_0 = \Gamma_{21}w_0 + \Gamma_{22}\varphi_0. \tag{2.452}$$

Vaaka- ja pystykomponenttien avulla

$$\delta_H = (\gamma_{11}\sin^2\alpha_0)H - (\gamma_{12}\sin\alpha_0)M_0, \tag{2.453}$$

$$\varphi_0 = -(\gamma_{21}\sin\alpha_0)H + \gamma_{22}M_0, \tag{2.454}$$

missä on käytetty yhteyksiä $\delta_H = w_0\sin\alpha_0 + u_0\cos\alpha_0 \approx w_0\sin\alpha_0$, $N_\alpha = \cot\alpha Q_\alpha$ ja

$$H = Q_0/\sin\alpha_0, \tag{2.455}$$

$$H = \left(\frac{\Gamma_{11}}{\sin^2\alpha_0}\right)\delta_H + \left(\frac{\Gamma_{12}}{\sin\alpha_0}\right)\varphi_0, \tag{2.456}$$

$$M_0 = \left(\frac{\Gamma_{21}}{\sin\alpha_0}\right)\delta_H + \Gamma_{22}\varphi_0. \tag{2.457}$$

Kuoren reunan läheisyydessä reunahäiriötehtävän avulla lasketut suureet

$$w(\psi) = \gamma_{11}e^{-\gamma\psi}\left[\left(Q_0 - \frac{K}{R_\beta}M_0\right)\cos\gamma\psi + \frac{K}{R_\beta}M_0\sin\gamma\psi\right], \tag{2.458}$$

$$\varphi(\psi) = \gamma_{12}e^{-\gamma\psi}\left[\left(-Q_0 + 2\frac{K}{R_\beta}M_0\right)\cos\gamma\psi - Q_0\sin\gamma\psi\right], \tag{2.459}$$

$$M_\alpha(\psi) = e^{-\gamma\psi}\left[M_0\cos\gamma\psi - \left(\frac{R_\beta}{K}Q_0 - M_0\right)\sin\gamma\psi\right], \tag{2.460}$$

$$Q_\alpha(\psi) = e^{-\gamma\psi}\left[Q_0\cos\gamma\psi + \left(-Q_0 + 2\frac{K}{R_\beta}M_0\right)\sin\gamma\psi\right]. \tag{2.461}$$

Reunasiirtymien avulla

$$w(\psi) = e^{-\gamma\psi}\left[w_0\cos\gamma\psi + \left(w_0 + \frac{R_\beta}{K}\varphi_0\right)\sin\gamma\psi\right], \tag{2.462}$$

$$\varphi(\psi) = e^{-\gamma\psi}\left[\varphi_0\cos\gamma\psi - \left(2\frac{K}{R_\beta}w_0 + \varphi_0\right)\sin\gamma\psi\right], \tag{2.463}$$

$$M_\alpha(\psi) = -\Gamma_{12}e^{-\gamma\psi}\left[-\left(w_0 + \frac{R_\beta}{K}\varphi_0\right)\cos\gamma\psi + w_0\sin\gamma\psi\right], \tag{2.464}$$

$$Q_\alpha(\psi) = \Gamma_{11}e^{-\gamma\psi}\left[\left(w_0 + \frac{R_\beta}{2K}\varphi_0\right)\cos\gamma\psi + \frac{R_\beta}{2K}\varphi_0\sin\gamma\psi\right], \tag{2.465}$$

$$\varepsilon_\beta = \frac{w}{R_\beta}, \quad N_\beta = \frac{Eh}{R_\beta}w. \tag{2.466}$$

2.9.1 Kalvotilan siirtymät

$$w_K = R_\beta\varepsilon_\beta, \quad \varphi_K = -\cot\alpha_0(\varepsilon_\beta - \varepsilon_\alpha) - \frac{R_\beta}{R_\alpha}\frac{d\varepsilon_\beta}{d\alpha}, \tag{2.467}$$

$$w_K = \frac{R_\beta}{Eh}(N_\beta - \nu N_\alpha), \tag{2.468}$$

$$\varphi_K = -\cot\alpha_0\frac{1+\nu}{Eh}(N_\beta - N_\alpha) - \frac{1}{Eh}\frac{R_\beta}{R_\alpha}\left(\frac{dN_\beta}{d\alpha} - \nu\frac{dN_\alpha}{d\alpha}\right). \tag{2.469}$$

Erikoistapauksia

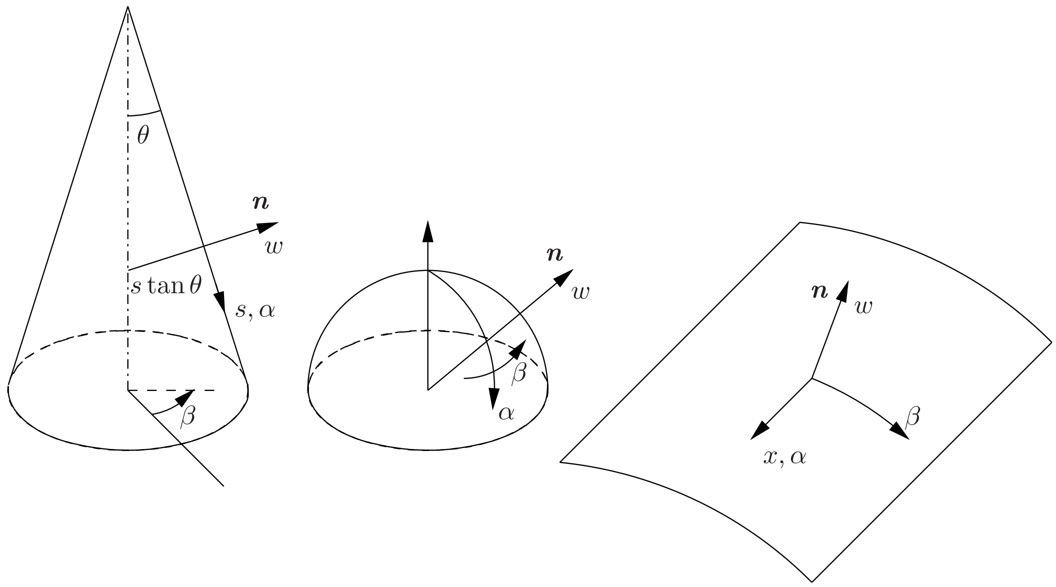
1. Sylinterikuori

$$R_\beta = a, \quad \frac{1}{R_\alpha}\frac{d}{d\alpha} \to \frac{d}{dx}, \tag{2.470}$$

$$w_K = \frac{a}{Eh}(N_\beta - \nu N_x) = \frac{a}{Eh}(af_n - \nu N_x), \tag{2.471}$$

$$\varphi_K = -\frac{a}{Eh}\left(\frac{dN_\beta}{dx} - \nu\frac{dN_x}{dx}\right) = -\frac{a}{Eh}\left(a\frac{df_n}{dx} - \nu\frac{dN_x}{dx}\right) \tag{2.472}$$

reunalla $x = x_0$.



Kuva 2.26 Kartio- pallo- ja lieriökuoren koordinaatistot.

2. Kartion reunalla $s = s_0$

$$w_K = \frac{s\tan\theta}{Eh}(N_\beta - \nu N_s), \tag{2.473}$$

$$\varphi_K = -\tan\theta\frac{1+\nu}{Eh}(N_\beta - N_s) - \frac{s\tan\theta}{Eh}\left(\frac{dN_\beta}{ds} - \nu\frac{dN_s}{ds}\right). \tag{2.474}$$

3. Pallokalotin ($R_\alpha = R_\beta$) reunalla $\alpha = \alpha_0$

$$w_K = \frac{a}{Eh}(N_\beta - \nu N_\alpha), \tag{2.475}$$

$$\varphi_K = -\cot\alpha_0\frac{1+\nu}{Eh}(N_\beta - N_\alpha) - \frac{1}{Eh}\left(\frac{dN_\beta}{d\alpha} - \nu\frac{dN_\alpha}{d\alpha}\right). \tag{2.476}$$

2.9.2 Yhteensopivuusyhtälöt

$$w_T + w_K = w_r, \tag{2.477}$$

$$\varphi_T + \varphi_K = \varphi_r, \tag{2.478}$$

$$\gamma_{11}Q_0 - \gamma_{12}M_0 + w_K = w_r, \tag{2.479}$$

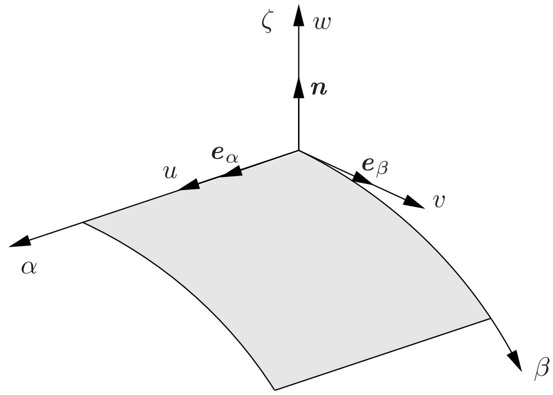
$$-\gamma_{21}Q_o + \gamma_{22}M_o + \varphi_K = \varphi_r, \tag{2.480}$$

w_r ja φ_r tuen siirtymä ja kiertymä.

2.10 Sylinterikuori

Tasapainoehdot, (muuttujanvaihdos $s = a\beta$, $ds = ad\beta$)

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{sx}}{\partial s} + af_x = 0, \tag{2.481}$$



Kuva 2.27 Sylinterikuoren koordinaatisto.

$$\frac{\partial N_s}{\partial s} + \frac{\partial N_{xs}}{\partial x} + \frac{1}{a} Q_\beta + f_s = 0, \tag{2.482}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_s}{\partial s} - \frac{1}{a} N_\beta + f_n = 0, \tag{2.483}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{sx}}{\partial s} - Q_x = 0, \tag{2.484}$$

$$\frac{\partial M_s}{\partial s} + \frac{\partial M_{xs}}{\partial x} - Q_s = 0. \tag{2.485}$$

Ohuen sylinterikuoren muodonmuutokset

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + \zeta \boldsymbol{\kappa}, \tag{2.486}$$

keskipinnan muodonmuutosvektori ja käyristymävektori

$$\boldsymbol{\varepsilon}_0 = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial s} + \dfrac{w}{a} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial s} \end{bmatrix}, \quad \boldsymbol{\kappa} = \begin{bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial s^2} + \underline{\dfrac{1}{a}\dfrac{\partial v}{\partial s}} \\ -2\dfrac{\partial^2 w}{\partial x \partial s} + \underline{\dfrac{1}{a}\dfrac{\partial v}{\partial x}} \end{bmatrix}. \tag{2.487}$$

Yleistetty Hooken laki, kalvovoimat ja momentit

$$N_x = C(\varepsilon_x + \nu \varepsilon_s) = C\left[\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial s} + \frac{w}{a}\right)\right], \tag{2.488}$$

$$N_s = C(\varepsilon_s + \nu \varepsilon_x) = C\left(\frac{\partial v}{\partial s} + \frac{w}{a} + \nu \frac{\partial u}{\partial x}\right), \tag{2.489}$$

$$N_{xs} = \frac{1-\nu}{2} C \gamma_{xs} = \frac{1-\nu}{2} C\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial s}\right), \tag{2.490}$$

$$M_x = D(\kappa_x + \nu \kappa_s) = D\left[-\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial}{\partial s}\left(-\frac{\partial w}{\partial s} + \frac{v}{\underline{a}}\right)\right], \tag{2.491}$$

$$M_s = D(\kappa_s + \nu \kappa_x) = D\left[\frac{\partial}{\partial s}\left(-\frac{\partial w}{\partial s} + \frac{v}{\underline{a}}\right) - \nu \frac{\partial^2 w}{\partial x^2}\right], \tag{2.492}$$

$$M_{xs} = (1-\nu)D\kappa_{xs} = (1-\nu)D\frac{\partial}{\partial x}\left(-\frac{\partial w}{\partial s} + \frac{v}{2a}\right), \tag{2.493}$$

$$C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}. \tag{2.494}$$

Eliminoimalla leikkausvoimat

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{sx}}{\partial s} + af_x = 0, \tag{2.495}$$

$$\frac{\partial N_s}{\partial s} + \frac{\partial N_{xs}}{\partial x} + \frac{1}{a}\frac{\partial M_s}{\partial s} + \frac{1}{a}\frac{\partial M_{xs}}{\partial x} + f_s = 0, \tag{2.496}$$

$$-\frac{1}{a}N_s + \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xs}}{\partial x \partial s} + \frac{\partial^2 M_s}{\partial s^2} + f_n = 0. \tag{2.497}$$

Lausumalla jännitysresultantit siirtymien avulla

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial s^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial s} + \nu\frac{1}{a}\frac{\partial w}{\partial x} + \frac{1}{C}f_x = 0, \tag{2.498}$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial s} + \left(1 + \frac{h^2}{12a^2}\right)\left(\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial s^2}\right) + \frac{1}{a}\left(\frac{\partial w}{\partial s} - \frac{h^2}{12}\frac{\partial}{\partial s}\nabla^2 w\right) + \frac{1}{C}f_s = 0, \tag{2.499}$$

$$\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a}\left(\frac{\partial v}{\partial s} - \frac{h^2}{12}\frac{\partial}{\partial s}\nabla^2 w\right) + \frac{1}{a^2}w + \frac{h^2}{12}\frac{\partial}{\partial s}\nabla^4 w - \frac{1}{C}f_n = 0, \tag{2.500}$$

$$\nabla^2(\bullet) = \frac{\partial}{\partial x^2}(\bullet) + \frac{\partial}{\partial s^2}(\bullet), \quad \nabla^4(\bullet) = \nabla^2(\bullet)\nabla^2(\bullet). \tag{2.501}$$

Pyörähdyssymmetrinen kuormitus

$$f_s = 0, \quad Q_s = N_{xs} = M_{xs} = v = 0, \tag{2.502}$$

tasapainoyhtälöt

$$\frac{d^2 u}{dx^2} + \nu\frac{1}{a}\frac{dw}{dx} + \frac{1}{C}f_x = 0, \tag{2.503}$$

$$\nu\frac{du}{dx} + \frac{w}{a} + a\frac{h^2}{12}\frac{d^4 w}{dx^4} - \frac{a}{C}f_n = 0. \tag{2.504}$$

Tapauksessa $f_x = 0$

$$\frac{du}{dx} + \frac{\nu}{a}w = \text{vakio} = C_5. \tag{2.505}$$

Hooken laki

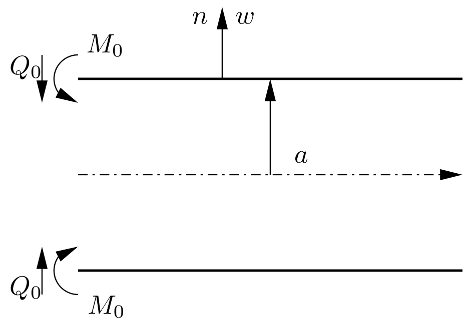
$$\frac{du}{dx} + \frac{\nu}{a}w = \frac{1}{C}N_x \quad \Rightarrow \quad \nu\frac{du}{dx} = \frac{\nu N_x}{C} - \frac{\nu^2 w}{a} \tag{2.506}$$

$$\Rightarrow \frac{d^4 w}{dx^4} + 4\lambda^4 w = \frac{1}{D}\left(f_n - \frac{\nu}{a}N_x\right), \quad \lambda^4 = \frac{3(1-\nu^2)}{a^2h^2}, \tag{2.507}$$

ratkaisu

$$w = e^{-\lambda x}(C_1\cos\lambda x + C_2\sin\lambda x) + e^{\lambda x}(C_3\cos\lambda x + C_4\sin\lambda x) + w_p(x), \tag{2.508}$$

$$u = C_6 + C_5 x - \frac{\nu}{a}\int_0^x w\,dx. \tag{2.509}$$



Kuva 2.28 Päästä kuormitettu puoliääretön putki.

Lyhyelle sylinterille

$$w(x) = C_1 \sin \lambda x \sinh \lambda x + C_2 \sin \lambda x \cosh \lambda x + C_3 \cos \lambda x \sinh \lambda x + C_4 \cos \lambda x \cosh \lambda x. \tag{2.510}$$

Taipuma ja kiertymä reunan $x = 0$ leikkausvoimasta ja momentista

$$w(0) = -\frac{1}{2\lambda^3 D}(Q_0 + \lambda M_0), \tag{2.511}$$

$$\varphi(0) = -\frac{dw(0)}{dx} = -\frac{1}{2\lambda^2 D}(Q_0 + 2\lambda M_0). \tag{2.512}$$

2.11 Taitekuori

2.11.1 Kuormituksen jakaminen särmille

$$p_1 = \frac{\cos \alpha_2}{\sin(\alpha_1 - \alpha_2)} p_B, \tag{2.513}$$

$$p_2 = \frac{\cos \alpha_1}{\sin(\alpha_1 - \alpha_2)} p_B. \tag{2.514}$$

Palkin (levyn AB), kuva 2.31, ala– ja yläreunan venymät

$$\varepsilon_A = \frac{M_0}{EW} - \frac{4}{EA} T_A - \frac{2}{EA} T_B, \tag{2.515}$$

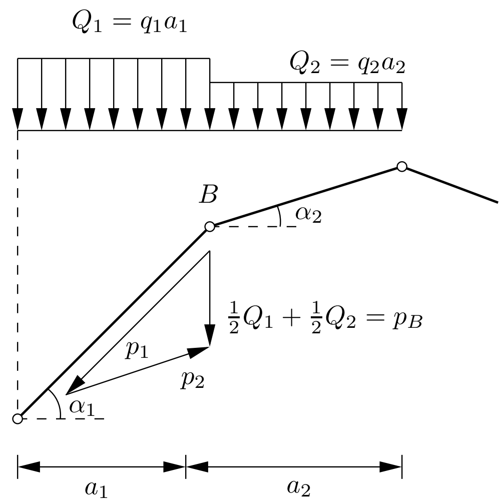
$$\varepsilon_B = -\frac{M_0}{EW} + \frac{2}{EA} T_A + \frac{4}{EA} T_B, \tag{2.516}$$

$$I = \frac{Ab^2}{12}, \quad A = bd, \quad W = \frac{db^2}{6}, \tag{2.517}$$

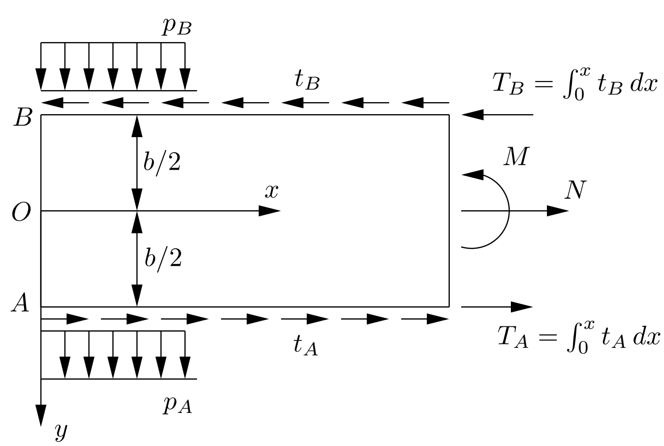
b on levyn leveys ja d on levyn paksuus. Yhteensopivusehto, kolmen leikkausvoiman yhtälö

$$\frac{2}{EA_1} T_A + \left(\frac{4}{EA_1} + \frac{4}{EA_2} \right) T_B + \frac{2}{EA_2} T_C = \frac{M_{10}}{EW_1} + \frac{M_{20}}{EW_2}, \tag{2.518}$$

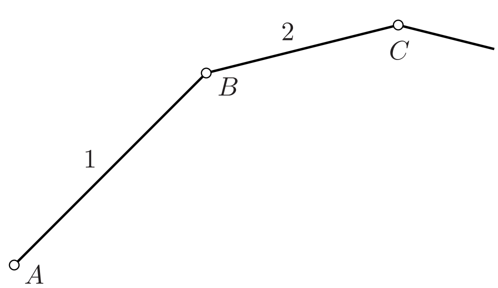
indeksit 1 ja 2 viittaavat levyihin.



Kuva 2.29 Kuormien jakaminen särmille.



Kuva 2.30 Taitekuoren osalevy.



Kuva 2.31 Levyjen ja särmien numerointi.

Luku 3

Fourier-muunnos

Parittoman funktion $f(x) = -f(-x)$ Fourier-sinimuunnos ja käänteismuunnos ovat

$$\begin{aligned} \bar{f}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) \sin \alpha\xi \, d\xi, \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}(\alpha) \sin \alpha x \, d\alpha. \end{aligned} \tag{3.1}$$

Muunnoksen edellytyksenä on, että

1. $f(x)$ on paloittain jatkuva jokaisella äärellisellä välillä,
2. $f(x)$ on absoluuttisesti integroituva eli

$$\int_{-\infty}^{\infty} |f(x)| dx < M < \infty. \tag{3.2}$$

Parillisen funktion $f(-x) = f(x)$ kosinimuunnos on vastaavasti

$$\begin{aligned} \bar{f}(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) \cos \alpha\xi \, d\xi, \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}(\alpha) \cos \alpha x \, d\alpha. \end{aligned} \tag{3.3}$$

Symmetrisen funktion f toisen derivaatan muunnos on

$$\bar{f}_{,xx}(\alpha) = -\alpha^2 \bar{f}(\alpha). \tag{3.4}$$

Taulukko 3.1 Fourier-sinimuunnoksia

$f(x)$	$\bar{f}(\alpha)$
$f=1,\quad 0<x<a,$ $f=0,\quad a<x<\infty$	$\sqrt{\frac{\pi}{2}}\frac{1-\cos\alpha a}{\alpha}$
$\frac{1}{x}$	$\sqrt{\frac{\pi}{2}}$
$\frac{x}{x^2+y^2},\quad \Re y>0$	$\sqrt{\frac{\pi}{2}}e^{-\alpha y}$
$\frac{2xy}{(x^2+y^2)^2},\quad \Re y>0$	$\sqrt{\frac{\pi}{2}}\alpha e^{-\alpha y}$
$\arctan\frac{x}{y},\quad \Re y>0$	$\sqrt{\frac{\pi}{2}}\frac{1}{\alpha}e^{-\alpha y}$
$\frac{1}{4}\ln\frac{(x+c)^2+y^2}{(x-c)^2+y^2}$	$\sqrt{\frac{\pi}{2}}\frac{1}{\alpha}e^{-\alpha y}\sin\alpha c$
$\frac{y}{(c-x)^2+y^2}-\frac{y}{(c+x)^2+y^2},$ $\Re y>0,\quad c+iy\notin\Re$	$\sqrt{2\pi}e^{-\alpha y}\sin\alpha c$
$\frac{1}{x}\sin\beta x,\quad \beta>0$	$\sqrt{\frac{\pi}{2}}\frac{1}{2}\ln\left\|\frac{\alpha+\beta}{\alpha-\beta}\right\|$
$\frac{\pi}{2}e^{-kx},\quad k>0$	$\sqrt{\frac{\pi}{2}}\frac{\alpha}{\alpha^2+k^2}$
$\frac{\pi}{4}(2-kx)e^{-kx},\quad k>0$	$\sqrt{\frac{\pi}{2}}\frac{\alpha^3}{(\alpha^2+k^2)^2}$
$\frac{\pi}{4}\frac{x}{k}e^{-kx},\quad k>0$	$\sqrt{\frac{\pi}{2}}\frac{\alpha}{(\alpha^2+k^2)^2}$
$\frac{\pi}{4}\frac{1}{k^4}[2-(2+kx)e^{-kx}],\ k>0$	$\sqrt{\frac{\pi}{2}}\frac{1}{\alpha(\alpha^2+k^2)^2}$
$\frac{\pi}{2}\frac{1}{k^2}(1-e^{-kx}),\quad k>0$	$\sqrt{\frac{\pi}{2}}\frac{1}{\alpha(\alpha^2+k^2)}$

Taulukko 3.2 Fourier-kosinimuunnoksia

$f(x)$	$\bar{f}(\alpha)$
$f=1,\quad 0<x<a,$ $f=0,\quad a<x<\infty$	$\sqrt{\frac{\pi}{2}}\frac{\sin\,\alpha a}{\alpha}$
$\frac{y}{x^2+y^2},\quad \Re y>0$	$\sqrt{\frac{\pi}{2}}e^{-\alpha y}$
$\frac{y^2-x^2}{(x^2+y^2)^2},\quad \Re y>0$	$\sqrt{\frac{\pi}{2}}\alpha e^{-\alpha y}$
$\frac{1}{2}\ln\left\|\frac{x^2+z^2}{x^2+y^2}\right\|,\quad y>0,\quad z>0$	$\sqrt{\frac{\pi}{2}}\frac{1}{\alpha}\left(e^{-\alpha y}-e^{-\alpha z}\right)$
$\frac{c-x}{(c-x)^2+y^2}+\frac{c+x}{(c+x)^2+y^2}$	$\sqrt{2\pi}e^{-\alpha y}\sin\alpha c,\quad \Re y>\|\Im c\|$
$\frac{y}{(c-x)^2+y^2}+\frac{y}{(c+x)^2+y^2}$	$\sqrt{2\pi}e^{-\alpha y}\cos\alpha c,\quad \Re y>\|\Im c\|$
$\frac{\pi}{2}\frac{1}{k}e^{-kx},\quad k>0$	$\sqrt{\frac{\pi}{2}}\frac{1}{\alpha^2+k^2}$
$\frac{\pi}{4}\frac{1}{k}(1-kx)e^{-kx},\quad k>0$	$\sqrt{\frac{\pi}{2}}\frac{\alpha^2}{(\alpha^2+k^2)^2}$
$\frac{\pi}{4}\frac{1}{k^3}(1+kx)e^{-kx},\quad k>0$	$\sqrt{\frac{\pi}{2}}\frac{1}{(\alpha^2+k^2)^2}$
$\frac{1}{2}\arctan\left(\frac{2}{x^2}\right)$	$\sqrt{\frac{\pi}{2}}\frac{1}{\alpha}e^{-\alpha}\sin\alpha$