

33001 Rakenteiden plastisuusmallit

Plastisuusteorian soveltaminen laattoihin

Ohuen laatan (Kirchhoffin) teorian johtamisessa otaksutaan, että:

1. Laatan taipuma w on pieni eli $w \ll h$ (h on laatan paksuus).
2. Laatan keskipinta ei veny.
3. Laatan keskitason normaalit säilyvät suorina ja keskitason normaaleina deformatiivisessa tilassa (Kirchhoffin otaksuma).

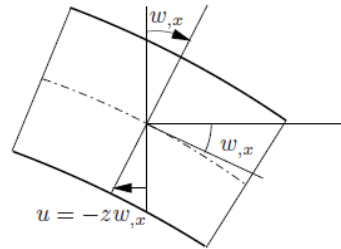
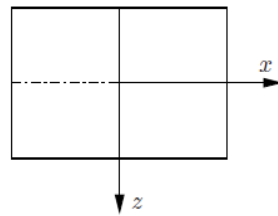
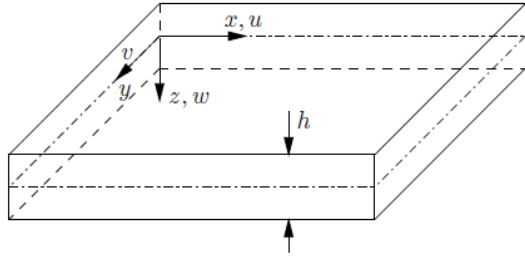
$$u = -zw_{,x} \equiv -z \frac{\partial w}{\partial x}$$

$$v = -zw_{,y} \equiv -z \frac{\partial w}{\partial y}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -zw_{,xx}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2zw_{,xy}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -zw_{,yy}$$



$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p = 0.$$

$$q_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}$$

$$q_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

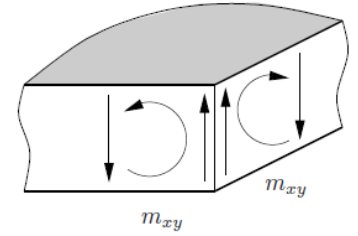
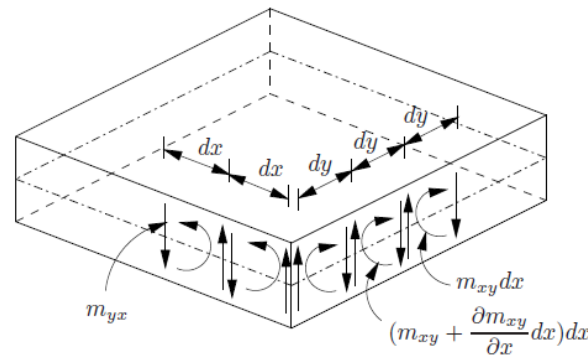
$$m_x = \int_{-h/2}^{h/2} z \sigma_x dz, \quad m_y = \int_{-h/2}^{h/2} z \sigma_y dz, \quad m_{xy} = \int_{-h/2}^{h/2} z \tau_{xy} dz.$$

$$q_x = \int_{-h/2}^{h/2} \tau_{xz} dz, \quad q_y = \int_{-h/2}^{h/2} \tau_{yz} dz$$

$$\frac{\partial^2 m_x}{\partial x^2} + 2 \frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + p = 0.$$

$$v_n = q_n + \frac{\partial m_{ns}}{\partial s}$$

$$v_n = q_n + \frac{\partial m_{ns}}{\partial s} = \frac{\partial m_n}{\partial n} + 2 \frac{\partial m_{ns}}{\partial s}$$



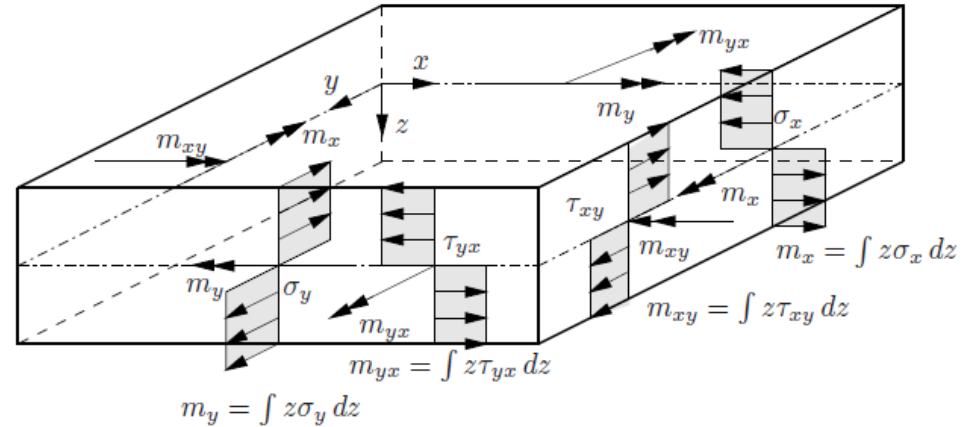
$x = \text{vakio}$

$$v_x = \frac{\partial m_x}{\partial x} + 2 \frac{\partial m_{xy}}{\partial y}$$

$y = \text{vakio}$

$$v_y = \frac{\partial m_y}{\partial y} + 2 \frac{\partial m_{xy}}{\partial x}$$

$$R = -2m_{xy}$$



$$m_x = \int_{-h/2}^{h/2} z \sigma_x dz = \sigma_x \frac{h^2}{4},$$

$$m_y = \int_{-h/2}^{h/2} z \sigma_y dz = \sigma_y \frac{h^2}{4},$$

$$m_{xy} = \int_{-h/2}^{h/2} z \tau_{xy} dz = \tau_{xy} \frac{h^2}{4}.$$

$$\sigma_{mx} = \sigma_{my} = \sigma_m$$

$$\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_m^2$$

von Mises:

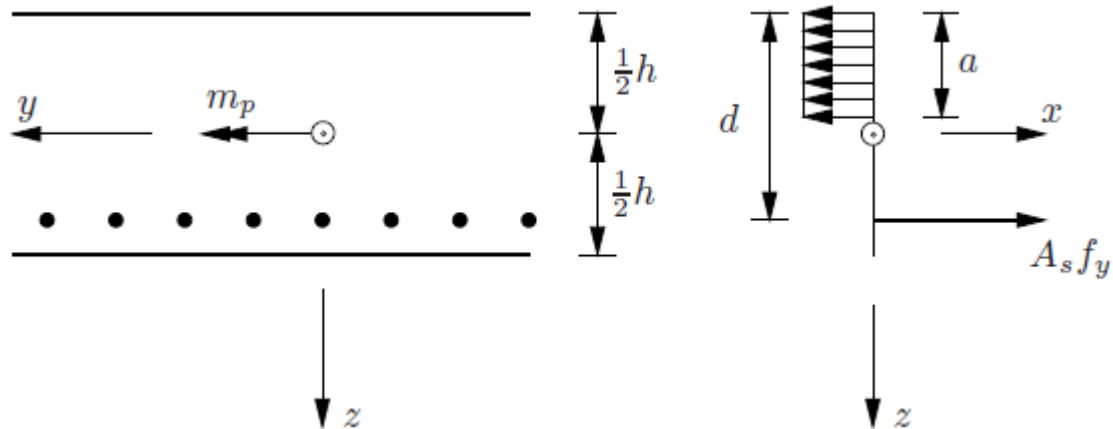
$$m_x^2 - m_x m_y + m_y^2 + 3m_{xy}^2 = m_p^2.$$

$$m_p = \sigma_m \frac{h^2}{4}$$

Tresca

$$\max\{|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|\} = \sigma_m$$

$$\max\{|m_1|, |m_2|, |m_1 - m_2|\} = m_p.$$



$$A_s f_y = a f_c$$

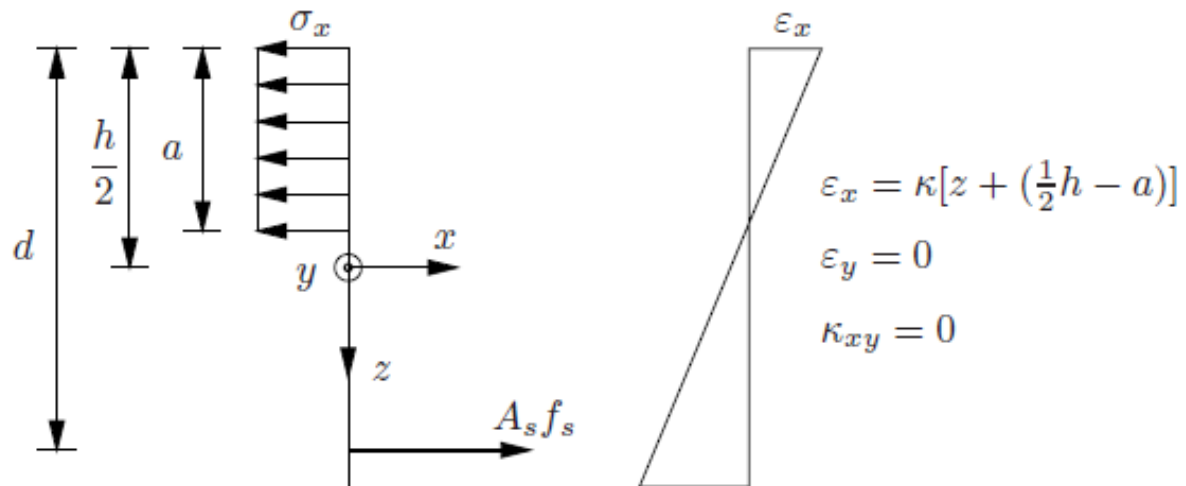
$$\Phi = \frac{A_s f_y}{d f_c}$$

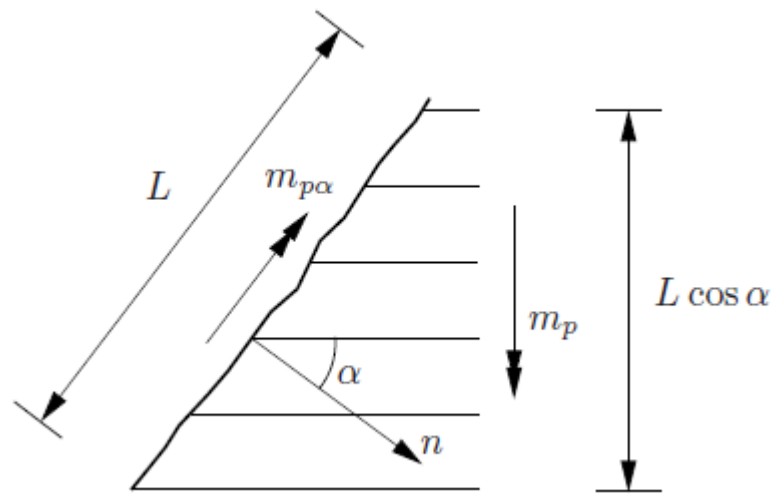
$$\frac{a}{d} = \Phi$$

$$m_p = A_s f_y \left(d - \frac{1}{2} a \right) = \left(1 - \frac{1}{2} \Phi \right) A_s f_y d = \left(1 - \frac{1}{2} \Phi \right) \Phi d^2 f_c$$

$$\varepsilon_x = \kappa \left[z + \left(\frac{1}{2} h - a \right) \right]$$

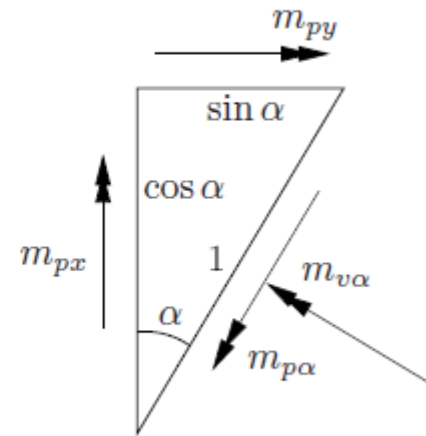
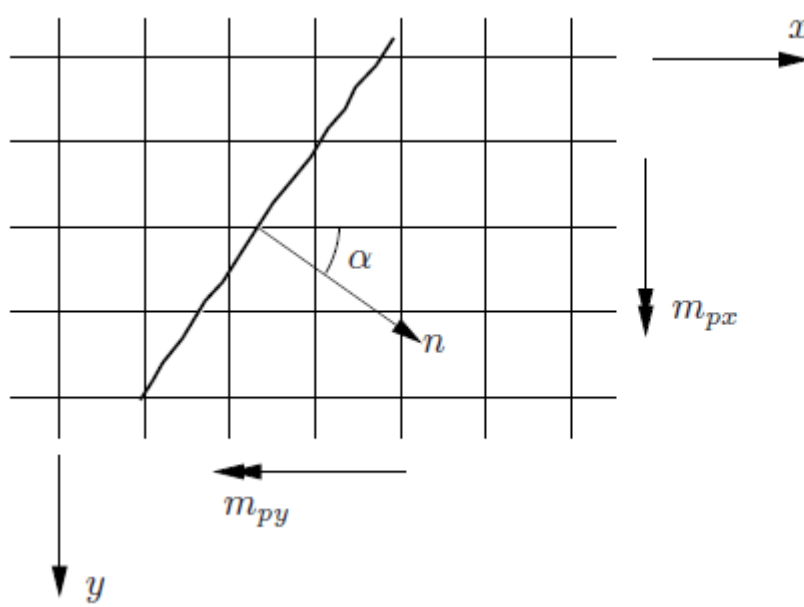
$$\varepsilon_y = 0 \quad \text{ja} \quad \gamma_{xy} = 0,$$





$$L m_{p\alpha} = m_p \cos \alpha L \cos \alpha$$

$$m_{p\alpha} = m_p \cos^2 \alpha.$$



$$m_{p\alpha} = m_{px} \cos^2 \alpha + m_{py} \cos^2 \left(\frac{\pi}{2} - \alpha \right)$$

$$= m_{px} \cos^2 \alpha + m_{py} \sin^2 \alpha,$$

$$m_{v\alpha} = (m_{py} - m_{px}) \sin \alpha \cos \alpha.$$

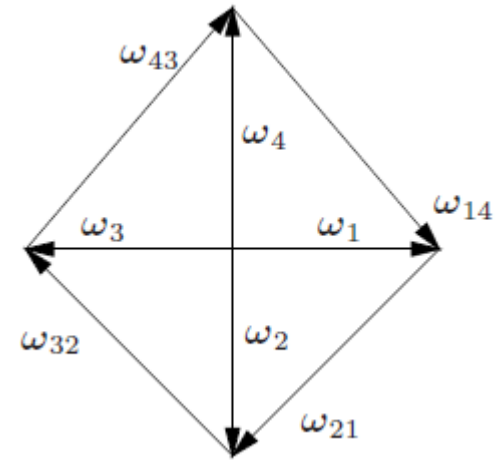
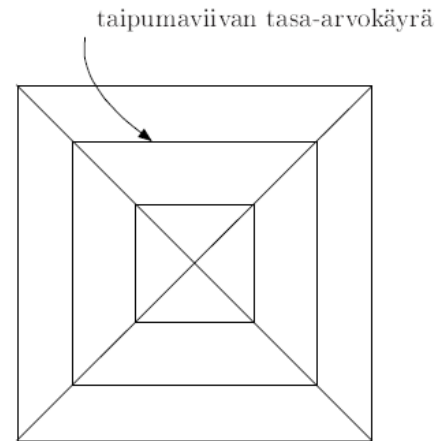
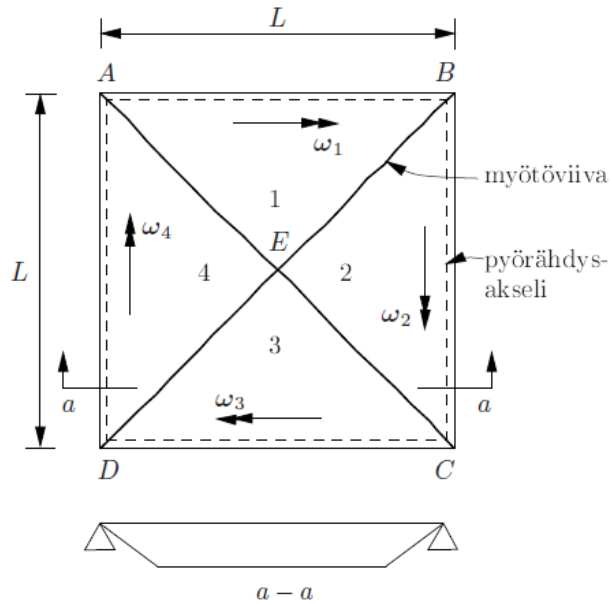
Myötöviivakuvioita koskevia sääntöjä

1. Myötöviivat ovat (tavallisesti) suorina ja ne ovat pyörähdysakseleita.
2. Myötöviiva loppuu laatan reunaan.
3. Pyörähdysakselit yhtyvät tuettuihin reunoihin, leikkaavat vapaat reunat ja kulkevat pilaritukien kautta.
4. Vierekkäisten laatan osien pyörähdysakselit leikkaavat toisensa (mahdollisesti ääretömyydessä).
5. Myötöviivat ovat positiivisia tai negatiivisia sen mukaan, mikä on momentin merkki.

Mekanismin kinematiikkaa

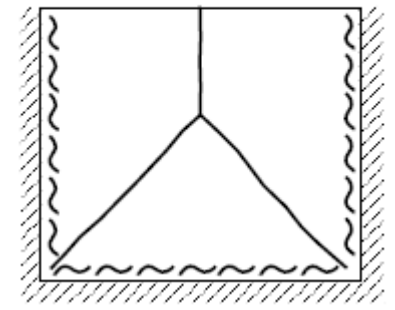
$$p = n - 1 + r$$

n = osa-alueiden lukumäärä
 r = kiertymisakselien
 parametrien lukumäärä

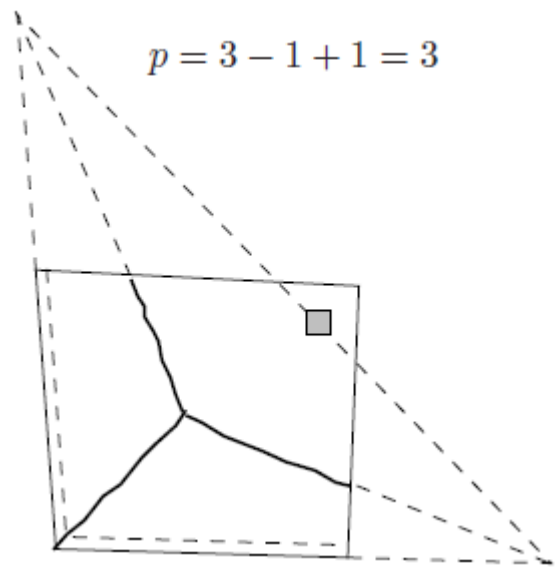


- kiertymisakseli
- ~~~~~ myötöviiva
- ~~~~~ negatiivinen myötöviiva
- =====- vapaasti tuettu reuna
- ====/====/====/ jäykästi tuettu reuna
- ===== vapaa reuna
- ===== palkki
- pilari

$$p = 3 - 1 + 0 = 2$$



$$p = 3 - 1 + 1 = 3$$



Esimerkki 15.1 Määritetään isotrooppisen, vapaasti tuetun suorakaidelaatan tasaisen kuorman yläraja-arvo.

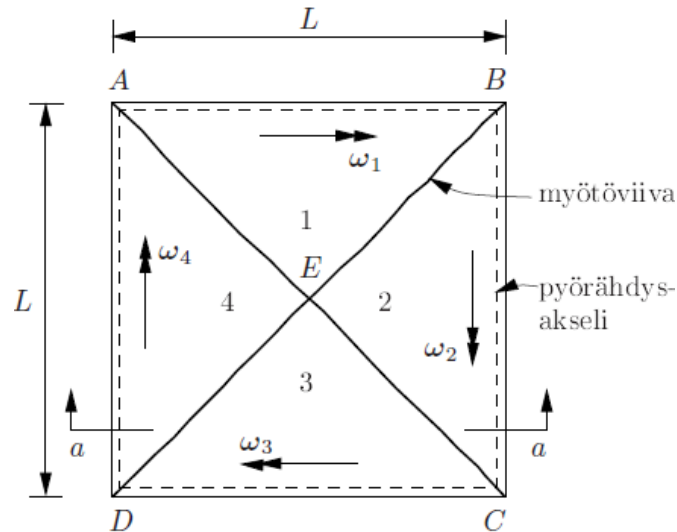
$$\omega_1 = \frac{w}{b/2}, \quad \omega_2 = \omega_3 = \omega_4 = \omega_1$$

$$\omega_{21} = \sqrt{2}\omega_1 = 2\sqrt{2}\frac{w}{b}$$

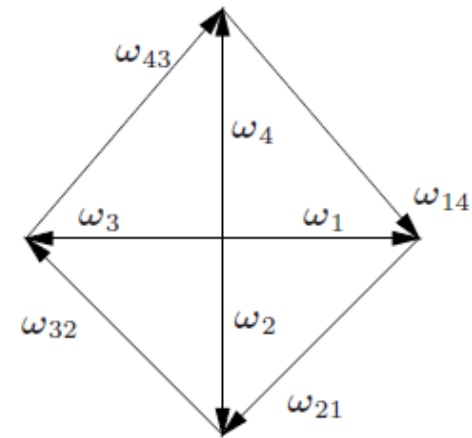
$$\omega_{13} = 2\omega_1 = 4\frac{w}{b}.$$

$$\begin{aligned} W_s &= m_p \left\{ 4\frac{w}{b}(a-b) + 2\sqrt{2}\frac{w}{b} \cdot 2\sqrt{2}b \right\} \\ &= 4m_p \left(1 + \frac{a}{b} \right) w, \end{aligned}$$

$$W_u = p_0 \left\{ \frac{1}{2}abw - 2 \cdot \frac{1}{3} \cdot \frac{1}{2}bw\frac{1}{2}b \right\} = p_0 \frac{1}{2}b^2 \left\{ \frac{a}{b} - \frac{1}{3} \right\} w.$$

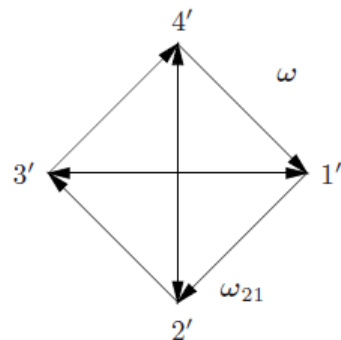
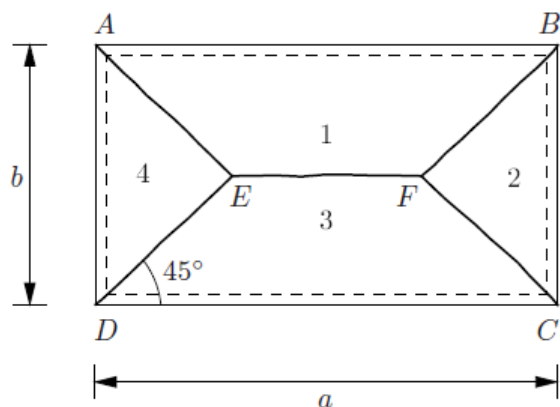


$$W_u = W_s$$



$$p_0^u = 8 \frac{m_p}{b^2} \frac{1 + \frac{b}{a}}{1 - \frac{1}{3} \frac{b}{a}}$$

Esimerkki 15.2 Vapaasti tuetulla ortotrooppisella teräsbetonisella suorakaidelaatalla on tasainen kuorma p . Määritetään plastisen rajakuorman yläraja-arvo.



$$W_{s1} = m_{py} a \omega_1 = m_{py} a \omega$$

$$W_{s3} = W_{s1}$$

$$W_{s2} = m_{px} b \omega_2 = m_{px} b \frac{b}{2x} \omega$$

$$W_s = 2(W_{s1} + W_{s2}) = \left\{ m_{py} 2a + m_{px} \frac{b^2}{x} \right\} \omega$$

$$W_u = p \left\{ \frac{1}{2}(a - 2x)b + 2 \cdot \frac{1}{3}xb \right\} \frac{b}{2} \omega = p \frac{b^2}{12} (3a - 2x) \omega$$

$$W_s = W_u \quad m_{px} \frac{b^2}{x} + m_{py} 2a = p \frac{b^2}{12} (3a - 2x)$$

$$\omega_1 \equiv \omega$$

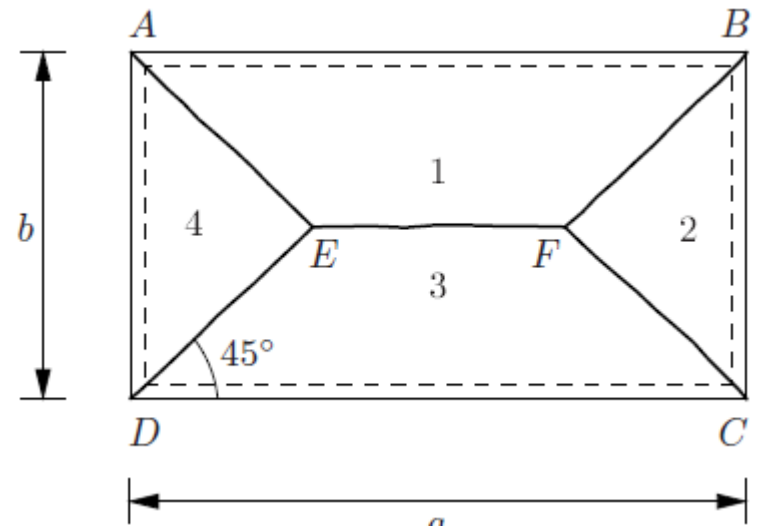
$$\omega_2 = \omega \tan \alpha = \omega_1 \frac{b}{2x}$$

$$\tan \alpha = \frac{b/2}{x}$$

$$p = \frac{12 m_{px} b^2 + 2 m_{py} a x}{b^2 x(3a - 2x)}$$

$$\frac{dp}{dx} = 0 \Rightarrow x = \left(\sqrt{1 + 3 \frac{a^2 m_{py}}{b^2 m_{px}}} - 1 \right) \frac{b^2 m_{px} a}{a^2 m_{py} 2}$$

$$p^u = \frac{\frac{24 m_{py} a^2 m_{py}}{b^2 b^2 m_{px}}}{\left\{ \sqrt{1 + 3 \frac{a^2 m_{py}}{b^2 m_{px}}} - 1 \right\}^2}$$



Jos $m_{px} = m_{py}$ ja $a/b = 2$, niin $x = (\sqrt{13} - 1) \frac{a}{8} \approx 0.3257a$, $\tan \alpha \approx 0.76759$, $\alpha \approx 37.51^\circ$

$$p^u = \frac{96 m_p / b^2}{\sqrt{13} - 1} \approx 14.1407 \frac{m_p}{b^2}$$

Esimerkki 15.3 Tarkastellaan isotrooppisesti raudoitettua neliölaattaa, jolla on tasainen kuorma p . Laatan kolme reunaa on vapaasti tuettu ja neljäs on vapaa. Määritetään rajakuorman yläraja-arvo myötöviivamenetelmällä.

Laatan osien 1, 2 ja 3 rotaatiot ovat

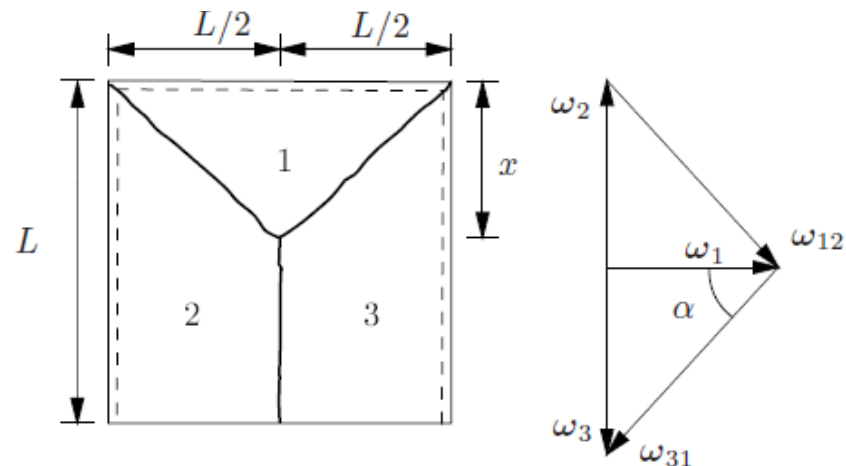
$$\omega_1 = \frac{1}{x}, \quad \omega_2 = \frac{2}{L} = \omega_3$$

$$W_s = m_p L \frac{1}{x} + 2m_p L \frac{2}{L}$$

$$W_u = \frac{1}{3} p \frac{Lx}{2} + 2p \left\{ (L-x) \frac{L}{2} \frac{1}{2} + \frac{xL/2}{2} \cdot \frac{1}{3} \right\}$$

$$W_s = W_u \quad m_p \left(\frac{L}{x} + 4 \right) = pL^2 \left(\frac{1}{2} - \frac{x}{6L} \right)$$

$$m_p = pL^2 \left\{ \frac{3 - \frac{x}{L}}{6 \left(\frac{L}{x} + 4 \right)} \right\}$$



Ehdosta $dm/dx = 0$ seuraa $x/L \approx 0.65$,

$$m_p \approx \frac{pL^2}{14.1}$$

