Introduction to materials modelling

Lecture 6 - Transversely isotropic elasticity

Reijo Kouhia

Tampere University, Structural Mechanics

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Constitutive models - symmetries

Eight possible linear elastic symmetries

Figure from Chadwick, Vianello, Cowin, *JMPJ*, 2001.
**Transverse isotropy**

**Definition:** For transversely isotropic material there exists a material direction defined by a unit vector $\mathbf{m}$ such that the constitutive relations are unchanged for arbitrary rotations of the coordinate system about that axis.

Examples: unidirectionally reinforced materials, stratified soils and rocks. Materials with hexagonal close packed crystal structure.

Figures: unidirectional fibres from mscsoftware.com and Grand Canyon by Luca Galuzzi
The specific strain energy (or alternatively the specific complementary energy) can depend on five invariants

$$W = W(I_1, I_2, I_3, I_4, I_5),$$

where the invariants can be defined as

$$I_1 = \text{tr}\varepsilon, \quad I_2 = \frac{1}{2} \text{tr}(\varepsilon^2), \quad I_3 = \frac{1}{3} \text{tr}(\varepsilon^3),$$

$$I_4 = \text{tr}(\varepsilon M), \quad I_5 = \text{tr}(\varepsilon^2 M),$$

where $M = mm^T$ is the structural tensor of transverse isotropy.
Linear transversely isotropic elasticity

**Five material parameters**, engineering constants

- $E_T$ Modulus of elasticity in the transverse isotropy plane,
- $E_L$ Modulus of elasticity in the longitudinal direction $m$,
- $G_L$ Shear modulus in the plane containing the symmetry axis,
- $\nu_T$ Poisson’s ratio in the isotropy plane,
- $\nu_L$ Poisson’s ratio in the isotropy plane when load in the longitudinal direction.
Linear transverse isotropy, restrictions to the elastic constants

Thermodynamic restrictions to the material parameters

\[ E_T > 0, \quad E_L > 0, \quad G_L > 0, \quad -1 < \nu_T < 1, \]

\[ -\sqrt{E_L/E_T} < \nu_L < \sqrt{E_L/E_T}, \]

\[ -\sqrt{\frac{E_L(1 - \nu_T)}{2E_T}} < \nu_L < -\sqrt{\frac{E_L(1 - \nu_T)}{2E_T}}. \]

In addition the monotonicity of the modulus of elasticity in an arbitrary direction requires

\[ G_L \leq \frac{E_L}{2(1 + \nu_L)}. \]
Stress in the longitudinal direction 1, i.e. $\sigma_{11}$, measure $\varepsilon_{11}, \varepsilon_{22} = \varepsilon_{33}$, then $E_1 = E_L = \sigma_{11}/\varepsilon_{11}$ and $\nu_L = \nu_{12} = \nu_{13} = -\varepsilon_{22}/\varepsilon_{11}$.

Stress in the transverse direction, i.e. $\sigma_{22}$, measure strain in the three perpendicular direction $\varepsilon_{11}, \varepsilon_{22}$ and $\varepsilon_{33}$, then $E_2 = E_T = \sigma_{22}/\varepsilon_{22}$, $\nu_{23} = -\varepsilon_{33}/\varepsilon_{22} = \nu_T$.

Shear in the 1-2 plane, then $G_{12} = G_L = \tau_{12}/\gamma_{12}$. Note $G_{12} = G_{13}$.

This test is not necessary. Shear in the isotropy plane, i.e. in the 2-3 plane. $G_{23} = \tau_{23}/\gamma_{23}$. Could also be obtained from $G_{23} = E_2/(1 + \nu_{23})$. 
Linear transverse isotropy - determination of material constants (cont’d)

Figure from http://nptel.ac.in/courses/101104010/lecture12/12_4.htm