Introduction to materials modelling

Lecture 3 - Balance equations

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Equilibrium

\[ B^* \sigma = f \quad \text{equilibrium} \]
\[ \sigma = C \varepsilon \quad \text{constitutive model} \]
\[ \varepsilon = Bu \quad \text{kinematical relation} \]
Equilibrium

There can be *infinite* number of stress states which satisfy equilibrium equations!

- **Statically determined** problem (isostatic): stresses can be determined uniquely from the equilibrium equations.

- **Statically indetermined problem** (hyperstatic): stresses *cannot* be determined uniquely from the equilibrium equations.
Examples of equilibrium equations

Axially loaded bar
\[- \frac{dN}{dx} = f.\]

Euler-Bernoulli beam model
\[- \frac{d^2 M}{dx^2} = q,\]

Kirchhoff plate model
\[- \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial^2 M_y}{\partial y^2} = q,\]
3D continuum

Force equilibrium equations for 3D continuum in coordinate invariant form

\[-\text{div } \sigma^T = \rho b \quad \text{or} \quad -\nabla \cdot \sigma^T = \rho b\]

In rectangular cartesian coordinate system

\[-\frac{\partial \sigma_{ji}}{\partial x_j} = \rho b_i \quad \text{or} \quad -\sigma_{ji,j} = \rho b_i\]

Moment equilibrium requires symmetry of the stress tensor!

Von Kármán notation

\[-\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z} = \rho b_x,\]
\[-\frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{zy}}{\partial z} = \rho b_y,\]
\[-\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \sigma_z}{\partial z} = \rho b_z.\]
3D continuum - cylindrical coordinates

Cylindrical coordinates $r, \theta, z$

\[
-\frac{\partial \sigma_r}{\partial r} - \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\partial \tau_{zr}}{\partial z} - \frac{1}{r} (\sigma_r - \sigma_\theta) = \rho b_r, \\
-\frac{\partial \tau_{r\theta}}{\partial r} - \frac{1}{r} \frac{\partial \sigma_{\theta}}{\partial \theta} - \frac{\partial \tau_{z\theta}}{\partial z} - \frac{2}{r} \tau_{r\theta} = \rho b_\theta, \\
-\frac{\partial \tau_{r z}}{\partial r} - \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} - \frac{\partial \sigma_z}{\partial z} - \frac{1}{r} \tau_{r z} = \rho b_z.
\]
Plane stress and strain

Plane stress - thin bodies - traction free plane
Plane strain - thick bodies - displacements restricted normal to the plane
Plane stress

There exist a traction free plane - normal $n$ - such that

$$t = \sigma^T n = 0.$$

For example, a disk in $(x, y)$-plane: thus $\sigma_z = \tau_{zx} = \tau_{zy} = 0$.

\[
\begin{align*}
- \frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} &= \rho b_x \\
- \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_y}{\partial y} &= \rho b_y
\end{align*}
\]
How to solve?

In linear elasticity one can use.

- Potential approach: e.g. Airy stress function in 2D, Helmholtz decomposition (scalar and vector potential) in 3D
- Numerical methods like FEM, BEM, FDM.

For more general cases only numerical solutions are mostly feasible.