Introduction to materials modelling

1. home exercise set, tensors, stress, strain

1. Are the following expressions written in the index notation correct in the Cartesian coordinate system. Motivate your answer. If they are correct write out the equations without index notation. You can also associate the indexes to the coordinates in the following way: \( x = x_1, y = x_2, z = x_3 \). In the expressions the permutation symbol is denoted as \( \epsilon_{ijk} \).

(a) \( \epsilon_{ijk} \epsilon_{irs} = \delta_{jr} \delta_{ks} - \delta_{js} \delta_{kk} \)
(b) \( v_k = (a_i c_i) b_k - (a_j b_j) c_k \)
(c) \( u_i = \frac{\partial \phi}{\partial x_i} + \epsilon_{ijk} \frac{\partial \psi_k}{\partial x_j} \)
(d) \( (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \)
(e) \( -\frac{\partial p}{\partial x_m} + (\lambda + \mu) \frac{\partial}{\partial x_m} \left( \frac{\partial v_k}{\partial x_k} \right) + \mu \frac{\partial^2 v_m}{\partial x_k \partial x_k} + \rho b_m = \rho \frac{d^2 v_m}{dt^2} \)
(f) \( \frac{\partial \mu_{ji}}{\partial x_j} + \rho c_i + \epsilon_{ijk} \sigma_{jk} - \rho \frac{d \ell_i}{dt} = 0 \).

2. Let \( A \) be a symmetric second-order tensor.

(a) Derive expressions for \( A^4 \) and \( A^{-1} \) expressed in terms of the invariants of \( A \) and its powers at most of order two.
(b) Derive the expression of the determinant of \( A \) expressed in terms of the invariants of \( A \). Remember that \( \text{tr} A, \text{tr}(A^2) \) and \( \text{tr}(A^3) \) are also invariants.
(c) Derive the expression for the derivative

\[
\frac{\partial J_3}{\partial A_{ij}}
\]

where \( J_3 \) is the cubic invariant of the deviator of \( A \).

3. Determine the principal stresses and the second and third deviatoric invariants \( J_2 = \frac{1}{2} s_{ij} s_{ij}, J_3 = \det s \) for the stress states described below. Calculate also the value of the Lode angle \( \theta \). The components which are not shown are zero.

(a) Uniaxial stress state \( \sigma_{11} = \sigma_0 \).
(b) Equibiaxial stress state \( \sigma_{11} = \sigma_{22} = \sigma_0 \).
(c) Shear \( \sigma_{11} = -\sigma_{22} = \sigma_0 \).
(d) Simple shear \( \sigma_{12} = \tau_0 \).
(e) Single normal stress and shear \( \sigma_{11} = \sigma_0, \sigma_{12} = \sigma_{13} = \tau_0 \).
(f) Shear \( \sigma_{12} = \sigma_{13} = \sigma_{23} = \tau_0 \).

Symbols \( \sigma_0 \) and \( \tau_0 \)

4. Determine the principal stresses and the principal directions of the previous cases (c), (d) ja (f).
5. The stress state is given by the stress tensor
\[
\sigma = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \sigma_0,
\]
where \( \sigma_0 \) is some arbitrary stress value.

(a) Determine deviatoric stress tensor \( \mathbf{s} \).

(b) Calculate the deviatoric invariants \( J_2 \) and \( J_3 \) and the von Mises stress \( \sigma_e = \sqrt{3J_2} \).

(c) Determine the principal stresses and the direction corresponding to the largest principal stress.

(d) Calculate the maximum shear stress and the normal to the plane where it occurs.

(e) Calculate the Lode angle \( \theta \) and draw the stress point in the deviatoric plane.

(f) Determine the traction vector \( \mathbf{t} \) on the plane having the normal \((1, 2, 2)^T\). Determine also the normal and shear components on that plane.

6. The measured values of three strain gauges are \( 2 \cdot 10^{-3}, -10^{-3} \) and \( 10^{-3} \) in directions \( 0^\circ, 30^\circ \) and \( 60^\circ \) with respect to the \( x_1 \) axis. Determine the strain state on the plane of the measurement.

7. Let us investigate the following deformation
\[
\begin{align*}
x_1 &= b_1 + X_1 + c_2 X_3 - c_3 X_2, \\
x_2 &= b_2 + X_2 + c_3 X_1 - c_1 X_3, \\
x_3 &= b_3 + X_3 + c_1 X_2 - c_2 X_1,
\end{align*}
\]
where \( b_i \) and \( c_i, i = 1, 2, 3, \) are arbitrary constants. Determine the deformation gradient \( \mathbf{F} \), the right Cauchy-Green deformation tensor \( \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \), and the Green-Lagrange strain tensor \( \mathbf{E} = \frac{1}{2} (\mathbf{C} - \mathbf{I}) \) and the infinitesimal strain tensor \( \mathbf{\epsilon} \). How do you interpret the results. Draw a deformed square in the \((1,2)\)-plane when \( b_1 = 2, b_2 = 3 \) and \( c_3 = 1 \), the rest are zero. Determine the exact value for the volume change from the deformation gradient and and compare it to the value obtained by using the infinitesimal strain tensor \( \mathbf{\epsilon} \). Calculate also the deformed length of a fiber originally having length \( L \) in the \( x_1 \) axis direction.

Return your answers in the Moodle system at latest on Wednesday 30.10.2019