An adaptive integration scheme in computational inelasticity

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MOTIVATION

SIMO & HUGHES, *Computational Inelasticity*, Remark 3.3.2.2 on page 125:

“The overall superiority of the radial return method relative to other return schemes is conclusively established in Krieg and Krieg [1977]; Schreyer, Kulak and Kramer [1979] and Yoder and Whirley [1984].”

WHY?
SCALAR MODEL PROBLEM

Maxwell creep model

\[ \dot{\varepsilon}^{\text{in}} = \gamma \left( \sigma / \sigma_{\text{ref}} \right) \]
\[ \dot{\sigma} = E (\dot{\varepsilon} - \dot{\varepsilon}^{\text{in}}) \]
\[ \dot{\sigma} + (E \gamma / \sigma_{\text{ref}}) \sigma = E \dot{\varepsilon} \]
\[ \dot{\sigma} + \lambda \sigma = f, \quad \sigma(0) = \sigma_0 \]
\[ \lambda = E \gamma / \sigma_{\text{ref}} \geq 0, \quad f = \eta \lambda \sigma_{\text{ref}}, \quad \eta = \dot{\varepsilon} / \gamma \]
AMPLIFICATION FACTOR \((f = 0 \text{ and } \lambda \text{ constant})\)

\[
A_{bE} = A_{dG(0)} = \frac{1}{1 + \lambda \Delta t}
\]

\[
A_{dG(1)} = A_{\text{IRKR2A-3}} = \frac{1 - \frac{1}{3}\lambda \Delta t}{1 + \frac{2}{3}\lambda \Delta t + \frac{1}{6}(\lambda \Delta t)^2}
\]

\[
A_{dG(2)} = A_{\text{IRKR2A-5}} = \frac{1 - \frac{2}{5}\lambda \Delta t + \frac{1}{20}(\lambda \Delta t)^2}{1 + \frac{3}{5}\lambda \Delta t + \frac{3}{20}(\lambda \Delta t)^2 + \frac{1}{60}(\lambda \Delta t)^3}
\]

\[
A_{dG(1) - \text{Lobatto}} = A_{\text{IRKL3C-2}} = \frac{1}{1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2}
\]
\[ dG(0) = BE \]
\[ dG(1) - \text{Lobatto} \]
\[ dG(2) \]

\[ dG(1) - \text{Lobatto} = \text{IRK}L3C \text{ method is the most accurate when } \Delta t \text{ is large enough !!!!} \]

bE also good when \( \lambda \Delta t > 15 \)
SOME REQUIREMENTS

An ideal integrator for inelastic constitutive models should be:

1. $L$-stable

2. For $\dot{y} + \lambda y = 0 \ (\lambda \text{ constant})$ the amplification factor should be
   (a) strictly positive
   (b) monotonous (convex)

Padé $(0, q)$-approximations of $\exp(-\lambda t)$ are positive and monotonous.
$dG(1)$-Lobatto = IRKL3C-2 = Padé-(0,2) for $\dot{\sigma} + \lambda \sigma = 0$
DISCONTINUOUS GALERKIN METHOD

\[ \dot{\sigma} = f(\sigma), \]

dG(q): find \( \sigma \) (polynomial of degree \( q \), as test functions \( \tau \)) such that

\[
\int_{I_n} \left( \dot{\sigma} - f(\sigma) \right) : \tau \ dt + [\sigma_n] : \tau_n^- = 0
\]
Find $\sigma_{n+1}^-$ such that

$$b_n(\dot{\sigma}, \tau) + \sigma_n^+ : \tau_n^+ = -c_n(\sigma, \tau) + \sigma_n^- : \tau_n^-$$

with the bilinear form

$$b_n(\sigma, \tau) = \int_{I_n} \sigma : \tau \, dt$$

and the semi-linear form

$$c_n(\sigma, \tau) = \int_{I_n} \lambda n : C^e : \tau \, dt$$
Case \( q = 0 \)

\[
\sigma_n^+ = -\int_{I_n} \lambda n : C^e \, dt + \sigma_n^- \quad \Rightarrow \quad \sigma_{n+1}^- = -\Delta t_n \lambda n : C^e + \sigma_{n+1}^{\text{trial},-}
\]

Backward Euler scheme

\[
\sigma_{n+1}^- = -\Delta t_n \lambda_{n+1} n_{n+1} : C^e + \sigma_{n+1}^{\text{trial},-}
\]

Case \( q \geq 1 \), linearisation:

\[
b_n(\Delta \dot{\sigma}, \tau) + c_{n,\sigma}(\Delta \sigma, \tau) = -b_n(\dot{\sigma}, \tau) - c_n(\sigma, \tau) - \sigma_n^+ : \tau_n^+ + \sigma_{n+1}^{\text{trial},-} : \tau_n^+
\]

where

\[
c_{n,\sigma}(\Delta \sigma, \tau) = \int_{I_n} \left\{ \Delta \sigma : \left[ \left( \frac{\partial n}{\partial \sigma} : \dot{\varepsilon} \right) \otimes (n : C^e) \right] : \tau + \lambda \Delta \sigma : \frac{\partial n}{\partial \sigma} : C^e : \tau \right\} \, dt
\]
UNIAXIAL EXAMPLES

\[
\dot{\sigma} = E \left[ \dot{\varepsilon} - f^* \exp \left( -\frac{Q}{R \theta} \right) \sinh^m \left( \frac{\sigma}{\sigma_r} \right) \right], \tag{1}
\]

\[
\theta(t) = \theta_0 + \Delta \theta \left( \frac{t}{t_{\text{max}}} \right), \quad \text{where} \quad t_{\text{max}} = \frac{\epsilon_{\text{max}}}{\dot{\varepsilon}}
\]

\[
\theta_0 = 293 \text{ K}, \quad \Delta \theta = 40 \text{ K}
\]

Binary near eutectic Sn40Pb solder:

\[
E = 33 \text{ GPa} \quad Q = 12 \text{ kcal/mol} \quad \nu = 0.3 \quad R = 2 \cdot 10^{-3} \text{ kcal/mol} \cdot \text{K} \quad \sigma_r = 20 \text{ MPa} \quad f^* = 10^5 \text{ s}^{-1} \quad m = 3.5
\]
Bar in uniaxial tension (strain rate $10^{-5}$ 1/s)
Pulsatile uniaxial straining, strain rate $\pm 10^{-5} \text{ s}^{-1}$. 
Relative error ($\dot{\epsilon} = 10^{-5} \, 1/s$)
CONCLUDING REMARKS

• Underintegrated dG(1) method produces a method equivalent to IRKL3C

• Asymptotically third order accuracy can be obtained by changing Lobatto quadrature to Gauss-Legendre

• More stable in the case of damage?