A time discontinuous Petrov-Galerkin method for the integration of inelastic constitutive equations

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OUTLINE

• Motivation

• Integration algorithms
  – Amplification factors of a model problem
  – Discontinuous Galerkin approach

• Example

• Concluding remarks
MOTIVATION

Simo & Hughes, *Computational Inelasticity*, Remark 3.3.2.2 on page 125:

“The overall superiority of the radial return method relative to other return schemes is conclusively established in Krieg and Krieg [1977]; Schreyer, Kulak and Kramer [1979] and Yoder and Whirley [1984].”

WHY ??
**SCALAR MODEL PROBLEM**

Maxwell creep model

\[ \dot{\epsilon}^{in} = \gamma (\sigma / \sigma_{ref}) \quad \dot{\sigma} = E (\dot{\epsilon} - \dot{\epsilon}^{in}) \]

\[ \dot{\sigma} + (E\gamma / \sigma_{ref}) \sigma = E\dot{\epsilon} \]

\[ \dot{y} + \lambda y = f, \quad y(0) = y_0 \]

\[ \lambda = E\gamma / \sigma_{ref} \geq 0, \quad f = \eta \lambda \sigma_{ref}, \quad \eta = \dot{\epsilon} / \gamma \]
AMPLIFICATION FACTOR ($f = 0$ and $\lambda$ constant)

\[
A_{bE} = A_{dG(0)} = \frac{1}{1 + \lambda \Delta t}
\]

\[
A_{dG(1)} = A_{IRKR2A-3} = \frac{1 - \frac{1}{3} \lambda \Delta t}{1 + \frac{2}{3} \lambda \Delta t + \frac{1}{6} (\lambda \Delta t)^2}
\]

\[
A_{dG(2)} = A_{IRKR2A-5} = \frac{1 - \frac{2}{5} \lambda \Delta t + \frac{1}{20} (\lambda \Delta t)^2}{1 + \frac{3}{5} \lambda \Delta t + \frac{3}{20} (\lambda \Delta t)^2 + \frac{1}{60} (\lambda \Delta t)^3}
\]

\[
A_{IRKL3C-2} = \frac{1}{1 + \lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2}
\]
IRK3L method is the most accurate when $\Delta t$ is large enough !!!!
BE also good when $\lambda \Delta t > 15$
SOME REQUIREMENTS

Ideal integrator for inelastic constitutive models should be:

1. \( L \)-stable

2. For \( \dot{y} + \lambda y = 0 \) (\( \lambda \) constant) the amplification factor should be
   (a) strictly positive
   (b) monotonous (convex)

Padé \((0, q)\)-approximations of \( \exp(-\lambda t) \) are positive and monotonous.
IRKL3C-2 = Padé-(0,2) for \( \dot{y} + \lambda y = 0 \)
QUESTION

Can we design a discontinuous Galerkin method with these properties ??

ANSWER

yes, if using discontinuous Petrov-Galerkin approach or underintegration

\[ \int_{t_n}^{t_{n+1}} (\dot{y} + \lambda y)w \, dt + [y_n]w_n^+ = \int_{t_n}^{t_{n+1}} fw \, dt \]

where \([y_n] = y_n^+ - y_n^-\)
PETROV-GALERKIN

\[ y = N_1 y_n^+ + N_2 y_{n+1}^-, \quad w = N_1^w \omega_1 + N_2^w \omega_2 \quad \text{and} \quad \lambda = N_1 \lambda_n + N_2 \lambda_{n+1} \]

\[
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix}^T \left[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
y_n^+ \\
y_{n+1}^-
\end{pmatrix}
+ (y_n^+ - y_n^-) \begin{pmatrix}
1 \\
0
\end{pmatrix}
- \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}
\right] = 0
\]

\[ A_{ij} = m_{ij} + k_{ij}, \quad m_{ij} = \int_{t_n}^{t_{n+1}} N_i^w \dot{N}_j \, dt, \quad k_{ij} = \int_{t_n}^{t_{n+1}} \lambda N_i^w N_j \, dt \]

\[ y_{n+1}^- = \frac{-A_{21}}{(1 + A_{11}) A_{22} - A_{12} A_{21}} y_n^- + \frac{(1 + A_{11}) f_2 - A_{21} f_1}{(1 + A_{11}) A_{22} - A_{12} A_{21}} \]

UNDERINTEGRATION

Using 2-point Lobatto integration for the dG(1)-scheme we get the two stage IRKL3C-method
MODEL PROBLEM

\[
\dot{y} + \lambda y = f, \quad y(0) = y_0
\]

\[
(f = \eta \lambda \sigma_{\text{ref}}, \quad \eta = \dot{\epsilon}/\gamma, \quad \lambda = E\gamma/\sigma_{\text{ref}})
\]

\[
\lambda(t) = \lambda_0 [1 - \phi + \phi \exp(-\beta t)]
\]

Two special cases

increasing diffusivity \( \phi = -1 \)
vanishing diffusivity \( \phi = 1 \), then \( \lambda \to 0 \) when \( t \to \infty \)
Increasing diffusivity (strain softening)

\[ \beta = 0.5 \lambda_0, \phi = -1.0, \eta = 2.0 \]
Vanishing diffusivity (strain hardening)

\[ \beta = 0.5 \lambda_0, \phi = 1.0, \eta = 2.0 \]
MATERIAL MODEL

\[ \dot{\varepsilon}^{\text{in}} = \frac{3}{2} \gamma n, \quad \text{where} \quad n = s/\bar{\sigma} \]

The scalar \( \bar{\sigma} \) is the equivalent stress

\[ \gamma = f^* \exp \left( \frac{-Q}{R \theta} \right) \sinh^m \left( \frac{\bar{\sigma}}{\sigma_y} \right) \]
Bar in uniaxial tension (strain rate $10^{-5}$ 1/s)

$$\theta(t) = \theta_0 \pm \Delta \theta(t/t_{\text{max}}), \quad \text{where} \quad t_{\text{max}} = \epsilon_{\text{max}}/\dot{\epsilon}$$

$$\theta_0 = 293\text{K}, \quad \Delta \theta = 40\text{K}$$

Binary near eutectic Sn40Pb solder:

$$E = 33\text{ GPa}$$

$$Q = 12\text{ kcal/mol}$$

$$\nu = 0.3$$

$$R = 2 \cdot 10^{-3}\text{ kcal/mol}\cdot\text{K}$$

$$\sigma_y = 20\text{ MPa}$$

$$f = 10^5\text{ s}^{-1}$$

$$m = 3.5$$
Thermally softening case \( \dot{\epsilon} = 10^{-5} \ 1/\text{s} \)
Thermally hardening case ($\dot{\epsilon} = 10^{-5} \text{ 1/s}$)
Relative error \( \dot{\varepsilon} = 10^{-5} \, 1/s \)

![Graph showing relative error vs \( \Delta t/t_{max} \)]
CONCLUDING REMARKS

- Two-stage IRKL3C method seems to be an accurate integrator also for large time steps
- Discontinuous Petrov-Galerkin approach can produce a method similar to IRKL3C
- Underintegrated dG(1) method produces a method similar to IRKL3C
- Asymptotically third order accuracy can be obtained by switching full integration in the dG(1) scheme if the time step is small