A geometric approach in computational stability analysis

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Dedicated to Niels
Outline

- Starting point
- The question of relevant buckling mode
- Stability eigenvalue problem
- Simple example
- Criticality manifold

Photograph Accident Investigation Board Finland
Collapse of Jyväskylä Fair Center roof
1st of Feb 2003
The questions

- What are the relevant buckling modes?
- Can they be adequately analysed by common computational tools?
- If not, how to analyse?
Stability eigenvalue problem

Definition for critical state: Find displacements \( q_{cr} \), critical load \( \lambda_{cr} \) and the corresponding eigenmode \( \phi \) such, that

\[
f'(q_{cr}, \lambda_{cr})\phi = 0 \quad \text{and} \quad f(q_{cr}, \lambda_{cr}) = 0,
\]

where \( f' = \partial f / \partial q \). The non-linear mapping \( f \) defines the equilibrium path in the displacement \( q \) and load parameter \( \lambda \) space:

\[
f(q, \lambda) \equiv r(q) - \lambda p_r(q) = 0
\]

and constitutes the balance between internal- and external forces.

System (1) is a non-linear eigenvalue problem, which is HARD TO SOLVE!
Polynomial eigenvalue problem

Expanding the nl-ev-problem into Taylor’s series wrt the state \((q_*, \lambda_*)\)

\[
q = q_* + \Delta \lambda q_1 + \frac{1}{2}(\Delta \lambda)^2 q_2 + \cdots
\]

results in a polynomial ev-problem:

\[
(K_{0|*} + \Delta \lambda K_{1|*} + \frac{1}{2}(\Delta \lambda)^2 K_{2|*} + \cdots) \phi = 0
\]

\[
K_{0|*} = f_*', \quad K_{1|*} = \frac{df}{d\lambda}|_{*} = f_*'' q_1 + f_*', \quad \dot{f} = \frac{\partial f}{\partial \lambda},
\]

\[
K_{2|*} = \frac{d^2 f}{d\lambda^2}|_{*} = f_*'' q_2 + f_*''' q_1 q_1 + 2f_*'' q_1 + \ddot{f}_*
\]
Linear stability eigenvalue problem

In the linear stability eigenvalue analysis the reference state is usually the initial state: \((q_*, \lambda_*) = (0, 0)\).\(^1\) The eigenvalue problem to be solved is

\[
(K_{0|0} + \lambda K_{1|0}) \phi = 0
\]

where the matrices are (assuming dead weight loading, i.e. \(\dot{f}' \equiv 0\))

\[
K_{0|0} = f'(0, 0), \quad K_{1|0} = f''(0, 0)q_1,
\]

and the pre-buckling displacement field, \(q_1\), is solved from \(K_{0|0}q_1 = p_r\).

Simple to solve, but CAN THE SOLUTION BE COMPLETELY USELESS!

\(^1\)This is usually the case in commercial FE software.
Simple example - state space

\[ \lambda_{\text{LIN}} = 2.0 \]
\[ \lambda_{\text{NL}} = 0.609 \]
\[ \frac{\lambda_{\text{NL}} - \lambda_{\text{LIN}}}{1 - |\langle \phi_{\text{LIN}}, \phi_{\text{NL}} \rangle|} = -2.281 \]
\[ |\langle \phi_{\text{LIN}}, \phi_{\text{NL}} \rangle| = 0.2442 \]
\[ \angle(\phi_{\text{LIN}}, \phi_{\text{NL}}) = 40.9 \, \text{deg} \]
Simple example - criticality manifold

$$\angle (\varphi(J_{\text{LIN}}), \varphi(J_{\text{NL}})) = 57.85 \text{ deg}$$

$$\ldots = \sqrt{2} \angle (\phi_{\text{LIN}}, \phi_{\text{NL}})$$
Error in the eigenmode
Concluding remarks

- Estimation of the error in the buckling mode
- Generalization of the error estimate to a general $n$-dimensional case
- Practical implications