On the choice of damage variable in the continuum fatigue model based on a moving endurance surface

Timo Saksala, Sami Holopainen, and Reijo Kouhia

Summary. This paper considers two different damage formulations for modelling high-cycle fatigue of materials. The underlying fatigue model is formulated within continuum mechanics framework with the concept of a moving endurance surface. Such a model has a unique feature that it allows for the concepts of fatigue limits and damage accumulation during the load history thus avoiding cycle-counting techniques. A scalar and tensor type of damage variables are utilized with an essentially similar type of damage evolution law. The tensor damage model capable of accounting for damage induced anisotropy is based on the gradient of the endurance surface. The performance of the scalar and tensor damage formulations are compared with different multidimensional stress histories.

Key words: high-cycle fatigue modelling, isotropic damage, anisotropic damage, endurance surface, evolution equations

Introduction

Fatigue of materials under variable loads is a complicated physical process which can even result in catastrophic failure of engineering components. It is characterized by nucleation, coalescence and stable growth of cracks. Nucleation of cracks starts from stress concentrations near persistent slip bands, grain interfaces and inclusions [1, 2, 3].

In high-cycle fatigue, the macroscopic behavior of the material is primarily elastic, while in the low-cycle fatigue regime considerable macroscopic plastic deformations take place. Transition between low- and high-cycle fatigue occurs between $10^3 - 10^4$ cycles. In recent years, it has been observed that fatigue failures can occur at very high fatigue lives $10^9 - 10^{10}$, below the previously assumed fatigue limits.

In this paper only high-cycle fatigue modelling is considered. Many different approaches have been proposed to model the high-cycle fatigue behaviour which can roughly be classified into stress invariant, or average stress based and critical plane approaches. In those approaches damage accumulation is usually based on cycle-counting, which makes their use questionable under complex load histories [4, 5].

A different strategy for high-cycle fatigue modelling was proposed by Ottosen et al. [4]. In their approach, which could be classified as evolutionary, the concept of a moving endurance surface in the stress space is postulated together with a damage evolution equation. The endurance surface is expressend in terms of the second invariant of the reduced deviatoric stress tensor where the center of the surface is defined by a deviatoric back stress tensor, as is done similarly in kinematic plasticity models. Therefore, the load history is memorized by the backstress tensor. In this model arbitrary stress states are treated in a unified manner for different loading histories, thus avoiding cycle-counting techniques.
In the present paper, different damage formulations to be used with the fatigue model by Ottosen et al. [4] are considered. Particularly, the original scalar damage formulation is compared with the proposed tensor damage model capable of accounting for damage induced anisotropy. Evolution of the tensorial damage variable is based on the normality condition for the endurance surface. Performance of the damage formulations are compared with some multidimensional stress histories.

**Model formulation**

**Endurance surface**

The continuum fatigue model developed in [4] is briefly described in the following. It is based on the assumption that a material exhibit loading condition dependent endurance limits within which no damage results under cyclic loading. Ottosen et al. [4] proposed a moving endurance surface in stress stress space to account for these limits. The endurance surface is of Drucker-Prager type as

$$\beta = \frac{1}{\sigma_{0e}} (\bar{\sigma} + AI_1 - \sigma_{0e}) = 0,$$

(1)

where $\sigma_{0e}$ is the endurance limit corresponding to zero mean stress, $A$ is a positive non-dimensional parameter, and $I_1 = \text{tr}(\sigma)$. In a constant amplitude cyclic, the endurance surface reduces to the linear relation in the Haigh diagram, i.e. relation between the mean stress and the stress amplitude, see Figure 1c. Moreover, $\bar{\sigma}$ in (1) is the effective stress defined in terms of the second invariant of the reduced deviatoric stress $s - \alpha$, with $\alpha$ being the back stress tensor, as

$$\bar{\sigma} = \sqrt{\frac{3}{2}} (s - \alpha) : (s - \alpha),$$

(2)

where $s = \sigma - \frac{1}{3} \text{tr}(\sigma) I$ is the deviatoric stress tensor, $I$ stands for the identity tensor, and

$$(s - \alpha) : (s - \alpha) := \text{tr}((s - \alpha)(s - \alpha))$$

(3)

is the double dot-product. The endurance surface, $\beta = 0$, moves in the stress space driven by the back stress which memorizes the load history. Contrarily to plasticity theory, the stress states out of the endurance surface, $\beta > 0$, are allowed. Moreover, the invariant $I_1$ in (1) accounts for the influence of the hydrostatic stress. The final model component needed before specifying the damage formulations is the evolution law for the back stress tensor. For this end, a hardening rule similar to Ziegler’s kinematic hardening rule in plasticity theory is adopted, i.e.

$$\dot{\alpha} = C(s - \alpha) \dot{\beta},$$

(4)

where $C$ is a non-dimensional material parameter, and the dot denotes time rate.

**Damage evolution**

In the original formulation by Ottosen et al. [4] a scalar damage variable is chosen to describe the material deterioration. Evolution equation for the damage $D$ used in [4] is

$$\dot{D} = K \exp(L\beta) \dot{\beta},$$

(5)

where $K$ and $L$ are parameters to be calibrated by experiments. From the evolution equation (5) it can be concluded that damage only develops when the stress state is moving away from the endurance surface, that is

$$\beta \geq 0, \quad \text{and} \quad \dot{\beta} > 0.$$  

(6)

It is also postulated that evolution for the back stress $\alpha$, see (4), takes place when the conditions (6) are satisfied. The conditions for evolution of damage and back stress in loading and unloading are illustrated in Figure 1a and b.
In this paper damage is described by a second order tensor, and therefore the model can account damage induced anisotropy. The evolution law for the proposed model is chosen to be of similar form as (5):

\[ \dot{D} = \dot{\beta}K_{\text{aniso}} \exp(L_{\text{aniso}}\beta) \left| \frac{\partial \beta}{\partial \sigma} \right|, \]  

where \( K_{\text{aniso}} \) and \( L_{\text{aniso}} \) are model parameters. Since damage never decreases the absolute value is taken of the gradient \( \partial \beta/\partial \sigma \).

In order to derive a stress-strain relationship, the specific strain energy function is postulated in the form

\[ W = \frac{1}{2} \lambda \left[ 1 - \frac{1}{3} \text{tr}(D) \right] \text{tr}(\epsilon)^2 + \mu \left[ \text{tr}(\epsilon^2) - \text{tr}(\epsilon^2D) \right], \]

where \( D \) is the symmetric damage tensor, and \( \lambda, \mu \) are the Lamé parameters. The stress-strain relationship is now obtained as a derivative of the specific strain energy function with respect to strain, i.e.

\[ \sigma = \frac{\partial W}{\partial \epsilon} = \lambda \left[ 1 - \frac{1}{3} \text{tr}(D) \right] \text{tr}(\epsilon)I + \mu(2\epsilon - \epsilon D - D \epsilon) = C_{\text{ed}} : \epsilon. \]

The fourth-order material secant stiffness tensor has the form

\[ C_{\text{ed}} = \lambda \left[ 1 - \frac{1}{3} \text{tr}(D) \right] I \otimes I + \mu(2I \otimes I - I \otimes D - D \otimes I), \]
where $\otimes$ denotes the standard tensor product, known as the Kronecker product, and the tensor product $\odot$ is defined as in Ref. [6]:

$$(A \odot B)_{ijkl} = \frac{1}{2}(A_{ik}B_{jl} + A_{il}B_{jk}).$$  \hfill (11)

A criterion for material failure is then provided by the requirement that the secant stiffness $C_{\text{ed}}$ should be positive definite. This can be checked e.g. by calculating its eigenvalues (which must be positive). The components of damage tensor $D$ can also be monitored but the interpretation of final failure is then more ambiguous.

**Numerical examples**

Some representative numerical simulations highlighting the capabilities of the anisotropic damage formulation above are presented in this section. The isotropic model calibration is the same as in [4]. Accordingly, the parameter values (for AISI-SAE 4340 alloy steel) are: $A = 0.025, \sigma_{0e} = 490 \text{ MPa}, C = 1.25, K = 2.65 \times 10^{-5}, L = 14.4$. The anisotropic damage evolution law is calibrated so that it matches the prediction of the isotropic model in the case of uniaxial alternating load (of sinus form) when the mean stress is zero and stress amplitude is 600 MPa. Due to the similarity of the damage evolution laws (5) and (7), the only change in parameter values needed is that $K_{\text{aniso}} = 2.32K$. Damage evolution for both damage formulations in uniaxial loading with some values of the stress amplitude and means stress are illustrated in Figure 2. In each case, the simulation is set to halt when secant stiffness $C_{\text{ed}}$ loses its positive definiteness. This criterion is implemented as the first normalized eigenvalue criterion $\lambda_1(t)/\lambda_1(t = 0) \geq 0$ (this quantity behaves similarly as the integrity variable, i.e. $1 - D$). The model comparisons

![Figure 2](image_url)

Figure 2. Damage evolution in uniaxial loading when (a) $\sigma_a = 600 \text{ MPa}, \sigma_m = 0$, (b) $\sigma_a = 700 \text{ MPa}, \sigma_m = 0$, (c) $\sigma_a = 600 \text{ MPa}, \sigma_m = 400 \text{ MPa}$, (d) and a part of load histories.
show that while a good agreement is obtained between the isotropic damage model and the anisotropic one with the first eigenvalue criterion in the case the zero mean stress (Figures 2a and b), a deviation of 6% occurs when $\sigma_m = 400\text{MPa}$. As for the first three diagonal components of the damage tensor, components $D_{22}$, $D_{33}$ evolve considerably despite the uni-axial loading. However, the final value of these components is only $1/3$ of that of the first component so that the model still accounts for the loading induced anisotropy. Moreover, the final value of component $D_{11}$ exceeds 1 in each case. Notwithstanding, the model could be calibrated so that the evolution of $D_{11}$ is identical to the evolution of isotropic damage in these load cases.

Next, multiaxial loading is considered. Namely, three special load histories leading to identical principal stress histories are tested. The first load case (LC1) has bi-axial pulsating normal stresses given by

$$\sigma_x = \sigma_a[1 + \sin(\omega t)], \quad (12a)$$
$$\sigma_y = \sigma_a[\sin(\omega t) - 1]. \quad (12b)$$

The second load case (LC2) has one pulsating normal stress and one pulsating shear stress as

$$\sigma_x = \sigma_a \sin(\omega t), \quad (13a)$$
$$\tau_{xy} = \frac{1}{2}\sigma_a \sin(\omega t - \pi/2). \quad (13b)$$

The third load case (LC3) has one pulsating normal stress and two pulsating shear stresses as

$$\sigma_x = \sigma_a \sin(\omega t), \quad (14a)$$
$$\tau_{xy} = \frac{1}{\sqrt{8}}\sigma_a \cos(\omega t), \quad (14b)$$
$$\tau_{yz} = -\frac{1}{\sqrt{8}}\sigma_a \cos(\omega t). \quad (14c)$$
Despite the identical principal stress histories, load cases LC1 and LC2 result in different endurance limits. For example, in LC1 steel 34Cr4 has endurance limit $\sigma_a = 240$ MPa while in LC2 the limit is $\sigma_a = 158$ MPa [7]. In the first test, the loading amplitude is $\sigma_a = 550$ MPa. With this amplitude, both damage formulations predicted immediate failure in LC1. Therefore, the behavior of the models is demonstrated with a lowered amplitude of $\sigma_a = 470$ MPa in LC1. The results are shown in Figure 3.

The results in Figure 3 display significant differences in the model predictions despite the fact that the loading histories have identical principal stress histories (equalling to LC1). In LC1 both models predict a huge initial jump in damage evolution after which the damage does not grow at all. The model behavior was similar with other values of stress amplitude. As for load cases LC2 and LC3, the isotropic damage model predicted identical damage evolution in them as can be observed in Figures 3b and c. In contrast, the anisotropic model with the eigenvalue criterion display a minor difference in the damage evolution so that the number cycles corresponding to failure is 115000 in LC2 and 120000 in LC3.

Conclusions

Different damage formulations in the continuum fatigue model based on a moving endurance surface by Ottosen et al. (2008) were very briefly considered in this study. The original scalar damage model was compared to a tensor damage model based on the gradient of the moving endurance surface. The latter model can account for loading induced anisotropy as was observed in the numerical simulations. According to this preliminary study, it seems that the anisotropic damage model with the eigenvalue criterion for final failure can - in contrast to the isotropic model - account for the influence of multiaxiality in load histories with identical principal stress histories. However, further studies are needed to state anything decisive on this issue.

Acknowledgements

This work has been supported by Tekes - the National Technology Agency of Finland, project SCarFace, decision number 40205/12.

References