Problem 1

Determine the derivative as a function of the global x-coordinate for the following quadratic isoparametric line element. Nodal coordinates are \( x_1 = 0, x_2 = \alpha L, x_3 = L \) (\( \alpha > 0 \)).

What is the allowable range of the parameter \( \alpha \)? The function to be interpolated is

\[
u(x) = u_3(x/L)^2 = \alpha^2 u_3 N_2 + u_3 N_3, \]

where \( N_2 = 1 - \xi^2, N_3 = \frac{1}{2} \xi (1 + \xi) \). Draw the derivative \( d\nu/dx \) with the following values of the \( \alpha \)-parameter: \( \alpha = 1/4 \) ja \( \alpha = 1/3 \). What can you say about the accuracy?

Solution

The function to be interpolated is \( u(x) = u_3(x/L)^2 \), thus the quadratic finite element interpolation

\[
\hat{u} = N_1 u_1 + N_2 u_2 + N_3 u_3,
\]

can exactly represent the given function. The quadratic interpolation functions are

\[
N_1 = \frac{1}{2} \xi (\xi - 1), \quad N_2 = 1 - \xi^2, \quad N_3 = \frac{1}{2} \xi (\xi + 1).
\]

Since \( u(x_1) = 0 \) and \( u(x_2) = \alpha^2 u_3 \) the FE-interpolant is

\[
\hat{u} = (\alpha^2 N_2 + N_3) u_3.
\]

Since the element is isoparametric, the geometry is also described with the same interpolation functions, i.e.

\[
x = N_1 x_1 + N_2 x_2 + N_3 x_3 = (\alpha N_2 + N_3) L.
\]

The derivative

\[
\frac{d\hat{u}}{dx} = \frac{1}{dx/d\xi} \frac{d\hat{u}}{d\xi} = J^{-1} \frac{d\hat{u}}{d\xi},
\]

where the Jacobian \( J \) is

\[
J = \frac{dx}{d\xi} = (\alpha N_2 \xi + N_3 \xi) = (-2\alpha \xi + \frac{1}{2} + \xi) L = [\frac{1}{2} + (1 - 2\alpha) \xi] L.
\]

The Jacobian has to be positive \( J > 0, \forall \xi \in (-1, 1) \), thus \( \frac{1}{4} < \alpha < \frac{3}{4} \).

The required derivative is now

\[
\frac{d\hat{u}}{dx} = \frac{1}{2} + (1 - 2\alpha^2) \xi \frac{u_3}{L} = \frac{1}{2} + (1 - 2\alpha^2) \xi \frac{u_3}{L},
\]

and using values

\[
\alpha = \frac{1}{3} \Rightarrow \frac{d\hat{u}}{dx} = \frac{1 + \frac{44}{9} \xi u_3}{1 + \frac{8}{3} \xi L}, \quad \text{and} \quad \alpha = \frac{1}{4} \Rightarrow \frac{d\hat{u}}{dx} = \frac{1 + \frac{7}{4} \xi u_3}{1 + \xi L}.
\]

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( x/L )</th>
<th>( L \frac{d\hat{u}}{dx}_u^3 )</th>
<th>( x/L )</th>
<th>( L \frac{d\hat{u}}{dx}_u^3 )</th>
<th>( x/L )</th>
<th>( L \frac{d\hat{u}}{dx}_u^3 )</th>
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</thead>
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<td>-5/3</td>
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<td>-\infty</td>
<td>-1/2</td>
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<td>1/2</td>
<td>5/8</td>
<td>5/4</td>
<td>9/16</td>
<td>4/3</td>
<td>1</td>
<td>11/8</td>
</tr>
</tbody>
</table>
In the figure below the derivative of the isoparametric element with unevenly spaced central node is shown. Also the exact derivative

\[
\frac{du}{dx} = \frac{2x u_3}{L \bar{L}}
\]

is shown. To draw the derivatives of the isoparametric element, the local \( \xi \)-coordinate has to be solved as a function of the global \( x \)-coordinate

\[
x = [\alpha(1 - \xi^2) + \frac{1}{2}\xi(1 + \xi)] L.
\]

For \( \alpha = \frac{1}{3} \) we get

\[
x/L = \frac{1}{3} + \frac{1}{2}\xi + \frac{1}{6}\xi^2,
\]

thus

\[
\xi = -\frac{3}{2} + \sqrt{\frac{1}{4} + 6(x/L)}.
\]

For \( \alpha = \frac{1}{4} \) we get

\[
x/L = \frac{1}{4} + \frac{1}{2}\xi + \frac{1}{4}\xi^2,
\]

thus

\[
\xi = -1 + 2\sqrt{x/L}.
\]
Problem 2
The nodal temperatures of an isoparametric element shown below are: \(u_1 = u_2 = u_5 = 0, u_3 = 2\bar{u}, u_4 = \bar{u}, u_6 = 5/8\bar{u}, u_7 = 35/16\bar{u}, u_8 = 1/2\bar{u}\). Assuming the material to be isotropic with thermal conductivity \(k\), determine the heat flux vector \(\vec{q} = -k\nabla u\) at node 4.

Solution
Geometry interpolation:
\[
\begin{align*}
x &= N_{2}^{\frac{1}{2}}L + N_{3}L + N_{5}^{\frac{1}{4}}L + n_{6}^{\frac{3}{4}}L + N_{7}^{\frac{1}{4}}L, \\
y &= N_{3}L + N_{4}L + n_{6}^{\frac{1}{2}}L + N_{7}^{\frac{3}{4}}L + N_{8}^{\frac{1}{4}}L.
\end{align*}
\]

Temperature in a similar way
\[
u = \sum_{i=1}^{8} N_{i}u_{i} = N_{3}2\bar{u} + 4\bar{u} + N_{6}^{\frac{5}{8}}\bar{u} + N_{7}^{\frac{35}{16}}\bar{u} + N_{8}^{\frac{1}{2}}\bar{u}.
\]

The interpolation functions are
\[
\begin{align*}
N_{2} &= \frac{1}{4}(1 + \xi)(1 - \eta)(\xi - \eta - 1), \\
N_{3} &= \frac{1}{4}(1 + \xi)(1 + \eta)(\xi + \eta - 1), \\
N_{4} &= \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1), \\
N_{5} &= \frac{1}{2}(1 - \xi^{2})(1 - \eta), \\
N_{6} &= \frac{1}{2}(1 - \eta^{2})(1 + \xi), \\
N_{7} &= \frac{1}{2}(1 - \xi^{2})(1 + \eta), \\
N_{8} &= \frac{1}{2}(1 - \eta^{2})(1 - \xi).
\end{align*}
\]
Thus
\[
x = \frac{1}{8}L(1 + \xi)(\eta + 3),
\]
\[
y = \frac{1}{8}L(1 + \eta)(5 - \xi^2),
\]
\[
u = \frac{1}{32}\bar{u}(1 + \eta)(29 + 2\xi + 6\eta(1 + \xi) - 11\xi^2),
\]

and the derivatives
\[
x_{,\xi} = \frac{1}{8}L(\eta + 3),
\]
\[
x_{,\eta} = \frac{1}{8}L(1 + \xi),
\]
\[
y_{,\xi} = \frac{1}{8}\xi(1 + \eta),
\]
\[
y_{,\eta} = \frac{1}{8}(5 - \xi^2),
\]
\[
u_{,\xi} = \frac{1}{32}\bar{u}(1 + \eta)(2 + 6 \xi - 22\xi),
\]
\[
u_{,\eta} = \frac{1}{32}\bar{u}(35 + 8 \xi + 12\eta + 12\xi\eta - 11\xi^2).
\]

The heat flux \( \vec{q} \) is
\[
\vec{q} = -k \left( \frac{\partial u}{\partial x} \vec{i} + \frac{\partial u}{\partial y} \vec{j} \right)
\]

To obtain the global derivatives we need the Jacobian
\[
\begin{bmatrix}
u_{,x} \\
\nu_{,y}
\end{bmatrix} = J^{-T} \begin{bmatrix}
u_{,\xi} \\
\nu_{,\eta}
\end{bmatrix}, \quad \text{where} \quad J^T = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\
x_{,\eta} & y_{,\eta}
\end{bmatrix}
\]

At node 4, \( \xi = -1 \) and \( \eta = 1 \), then
\[
J^T = \frac{L}{2} \begin{bmatrix} 1 & 1 \\
0 & 1
\end{bmatrix} \quad \Rightarrow \quad J^{-T} = \frac{2}{L} \begin{bmatrix} 1 & -1 \\
0 & 1
\end{bmatrix}.
\]

Also
\[
u_{,\xi}(-1,1) = \frac{15}{8}\bar{u}, \quad \text{and} \quad \nu_{,\eta}(-1,1) = \frac{1}{2}\bar{u}.
\]

Thus at node 4
\[
u_{,x} = \frac{2}{L}(u_{,\xi} - u_{,\eta}) = \frac{11}{4}\bar{u},
\]
\[
u_{,y} = \frac{2}{L}u_{,\eta} = \bar{u}/L,
\]

and finally
\[
\vec{q} = -(2\frac{37}{4} + 7)\frac{k\bar{u}}{L}.
\]