Problem 1

Determine the derivative as a function of the global x-coordinate for the following quadratic isoparametric line element. Nodal coordinates are $x_1 = 0, x_2 = \alpha L, x_3 = L$ $(\alpha > 0)$. What is the allowable range of the parameter α? The function to be interpolated is $u(x) = u_3(x/L)^2 = \alpha^2 u_3 N_2 + u_3 N_3$, where $N_2 = 1 - \xi^2, N_3 = \frac{1}{2}\xi(1+\xi)$. Draw the derivative du/dx with the following values of the α-parameter: $\alpha = 1/4$ ja $\alpha = 1/3$. What can you say about the accuracy?

Solution

The function to be interpolated is $u(x) = u_3(x/L)^2$, thus the quadratic finite element interpolation

$$\tilde{u} = N_1 u_1 + N_2 u_2 + N_3 u_3,$$

can exactly represent the given function. The quadratic interpolation functions are

$$N_1 = \tfrac{1}{2}\xi(\xi - 1), \quad N_2 = 1 - \xi^2, \quad N_3 = \tfrac{1}{2}\xi(\xi + 1).$$

Since $u(x_1) = 0$ and $u(x_2) = \alpha^2 u_3$ the FE-interpolant is

$$\tilde{u} = (\alpha_2^2 N_2 + N_3)u_3.$$

Since the element is isoparametric, the geometry is also described with the same interpolation functions, i.e.

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 = (\alpha N_2 + N_3)L.$$

The derivative

$$\frac{d\tilde{u}}{dx} = \frac{1}{\frac{dx}{d\xi}} \frac{d\tilde{u}}{d\xi} = J^{-1}\frac{d\tilde{u}}{d\xi},$$

where the Jacobian J is

$$J = \frac{dx}{d\xi} = (\alpha N_{2,\xi} + N_{3,\xi}) = (-2\alpha\xi + \tfrac{1}{2} + \xi)L = [\tfrac{1}{2} + (1 - 2\alpha)\xi]L.$$

The Jacobian has to be positive $J > 0, \forall \xi \in (-1, 1)$, thus $\frac{1}{4} < \alpha < \frac{3}{4}$.

The required derivative is now

$$\frac{d\tilde{u}}{dx} = \frac{\frac{1}{2} + (1 - 2\alpha^2)\xi}{\frac{1}{2} + (1 - 2\alpha)\xi} \frac{u_3}{L} = \frac{1 + 2(1 - 2\alpha^2)\xi}{1 + 2(1 - 2\alpha)\xi} \frac{u_3}{L},$$

and using values

$$\alpha = \tfrac{1}{3} \quad \Rightarrow \quad \frac{d\tilde{u}}{dx} = \frac{1 + \frac{14}{9}\xi}{1 + \frac{2}{3}\xi} \frac{u_3}{L}, \quad \text{and} \quad \alpha = \tfrac{1}{4} \quad \Rightarrow \quad \frac{d\tilde{u}}{dx} = \frac{1 + \frac{7}{4}\xi}{1 + \xi} \frac{u_3}{L}.$$

	$\alpha = \frac{1}{3}$		$\alpha = \frac{1}{4}$	
ξ	x/L	$\frac{L}{u_3}\frac{d\tilde{u}}{dx}$	x/L	$\frac{L}{u_3}\frac{d\tilde{u}}{dx}$
-1	0	-5/3	0	$-\infty$
-1/2	1/8	1/3	1/16	1/4
0	1/3	1	1/4	1
1/2	5/8	5/4	9/16	4/3
1	1	11/8	1	23/15

In the figure below the derivative of the isoparametric element with unevenly spaced central node is shown. Also the exact derivative

$$\frac{du}{dx} = 2\frac{x}{L}\frac{u_3}{L}$$

is shown. To draw the derivatives of the isoparametric element, the local ξ-coordinate has to be solved as a function of the global x-coordinate

$$x = \left[\alpha(1-\xi^2) + \tfrac{1}{2}\xi(1+\xi)\right] L.$$

For $\alpha = \frac{1}{3}$ we get

$$x/L = \tfrac{1}{3} + \tfrac{1}{2}\xi + \tfrac{1}{6}\xi^2,$$

thus

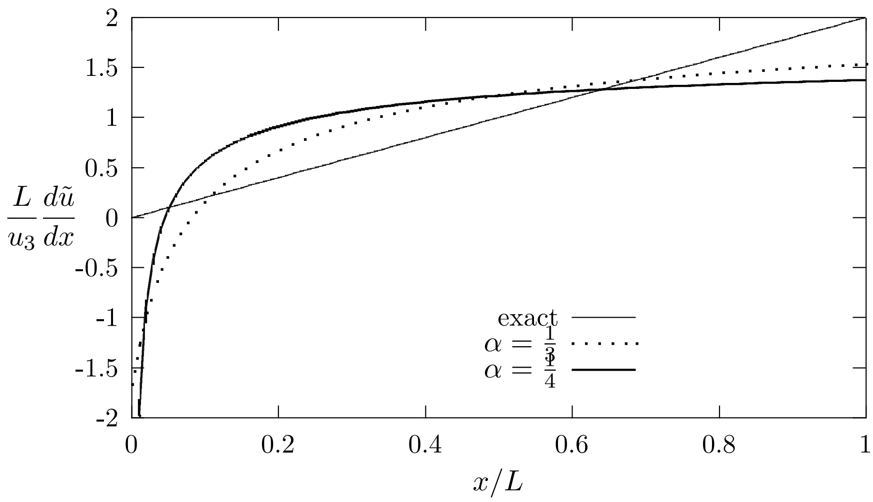
$$\xi = -\tfrac{3}{2} + \sqrt{\tfrac{1}{4} + 6(x/L)}.$$

For $\alpha = \frac{1}{4}$ we get

$$x/L = \tfrac{1}{4} + \tfrac{1}{2}\xi + \tfrac{1}{4}\xi^2,$$

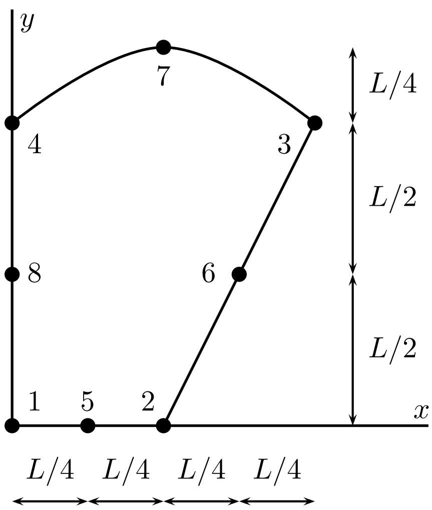
thus

$$\xi = -1 + 2\sqrt{x/L}.$$



Problem 2

The nodal temperatures of an isoparametric element shown below are: $u_1 = u_2 = u_5 = 0, u_3 = 2\bar{u}, u_4 = \bar{u}, u_6 = 5/8\bar{u}, u_7 = 35/16\bar{u}, u_8 = 1/2\bar{u}$. Assuming the material to be isotropic with thermal conductivity k, determine the heat flux vector $\vec{q} = -k\nabla u$ at node 4.



Solution

Geometry interpolation:

$$
\begin{aligned}
x &= N_2 \tfrac{1}{2}L + N_3 L + N_5 \tfrac{1}{4}L + n_6 \tfrac{3}{4}L + N_7 \tfrac{1}{2}L, \\
y &= N_3 L + N_4 L + n_6 \tfrac{1}{2}L + N_7 \tfrac{5}{4}L + N_8 \tfrac{1}{2}L.
\end{aligned}
$$

Temperature in a similar way

$$
u = \sum_{i=1}^{8} N_i u_i = N_3 2\bar{u} + 4\bar{u} + N_6 \tfrac{5}{8}\bar{u} + N_7 \tfrac{35}{16}\bar{u} + N_8 \tfrac{1}{2}\bar{u}.
$$

The interpolation functions are

$$
\begin{aligned}
N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1), \\
N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1), \\
N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1), \\
N_5 &= \tfrac{1}{2}(1-\xi^2)(1-\eta), \\
N_6 &= \tfrac{1}{2}(1-\eta^2)(1+\xi), \\
N_7 &= \tfrac{1}{2}(1-\xi^2)(1+\eta), \\
N_8 &= \tfrac{1}{2}(1-\eta^2)(1-\xi).
\end{aligned}
$$

Thus

$$
\begin{aligned}
x &= \tfrac{1}{8}L(1+\xi)(\eta+3),\\
y &= \tfrac{1}{8}L(1+\eta)(5-\xi^2),\\
u &= \tfrac{1}{32}\bar{u}(1+\eta)(29+2\xi+6\eta(1+\xi)-11\xi^2),
\end{aligned}
$$

and the derivatives

$$
\begin{aligned}
x_{,\xi} &= \tfrac{1}{8}L(\eta+3),\\
x_{,\eta} &= \tfrac{1}{8}L(1+\xi),\\
y_{,\xi} &= \tfrac{1}{4}\xi(1+\eta),\\
y_{,\eta} &= \tfrac{1}{8}(5-\xi^2),\\
u_{,\xi} &= \tfrac{1}{32}\bar{u}(1+\eta)(2+6\xi-22\xi),\\
u_{,\xi} &= \tfrac{1}{32}\bar{u}(35+8\xi+12\eta+12\xi\eta-11\xi^2).
\end{aligned}
$$

The heat flux $\vec{q}$ is

$$
\vec{q} = -k\left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)
$$

To obtain the global derivatives we need the Jacobian

$$
\left\{\begin{array}{c} u_{,x} \\ u_{,y} \end{array}\right\} = \boldsymbol{J}^{-T}\left\{\begin{array}{c} u_{,\xi} \\ u_{,\eta} \end{array}\right\}, \quad \text{where} \quad \boldsymbol{J}^T = \left[\begin{array}{cc} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{array}\right]
$$

At node 4, $\xi = -1$ and $\eta = 1$, then

$$
\boldsymbol{J}^T = \frac{L}{2}\left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right] \quad \Rightarrow \quad \boldsymbol{J}^{-T} = \frac{2}{L}\left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array}\right].
$$

Also

$$
u_{,\xi}(-1,1) = \tfrac{15}{8}\bar{u}, \quad \text{and} \quad u_{,\eta}(-1,1) = \tfrac{1}{2}\bar{u}.
$$

Thus at node 4

$$
\begin{aligned}
u_{,x} &= \frac{2}{L}(u_{,\xi} - u_{,\eta}) = \tfrac{11}{4}\frac{\bar{u}}{L},\\
u_{,y} &= \frac{2}{L}u_{,\eta} = \bar{u}/L,
\end{aligned}
$$

and finally

$$
\vec{q} = -(2\tfrac{3}{4}\vec{i} + \vec{j})\frac{k\bar{u}}{L}.
$$