Problem 1

For example, two problems where standard numerical schemes behave badly are in the stationary one-dimensional case: (a) the diffusion-convection equation and (b) the reaction-diffusion equation

$$-k\frac{d^2u}{dx^2} + \rho cv\frac{du}{dx} = 0, \tag{1a}$$

$$-k\frac{d^2u}{dx^2} + bu = 0, \quad \text{where} \quad b = \beta^2 k L^{-2} \tag{1b}$$

and β is a dimensionless parameter. It is assumed here that the physical parameters k, ρ, c, v, b are all constants in the domain $\Omega = \{x|x \in (0, L)\}$. Solve the problem with boundary conditions $u(0) = u_0 > 0, u(L) = 0$. Draw the solution with different values of the non-dimensional Péclet number $P = \rho cvL/k$, e.g. $P = 1, 10, 100$, and $\beta^2 = 1, 10, 100$. What happens when $P \to \infty$ and $\beta \to \infty$?

Solution for the case a: Let's try the solution in the form $u(x) = \exp(rx)$. Substituting it into the differential equation gives

$$(-kr^2 + \rho cvr)\exp(rx) = 0 \quad \Rightarrow \quad r = 0 \text{ or } r = \frac{\rho cv}{k}.$$

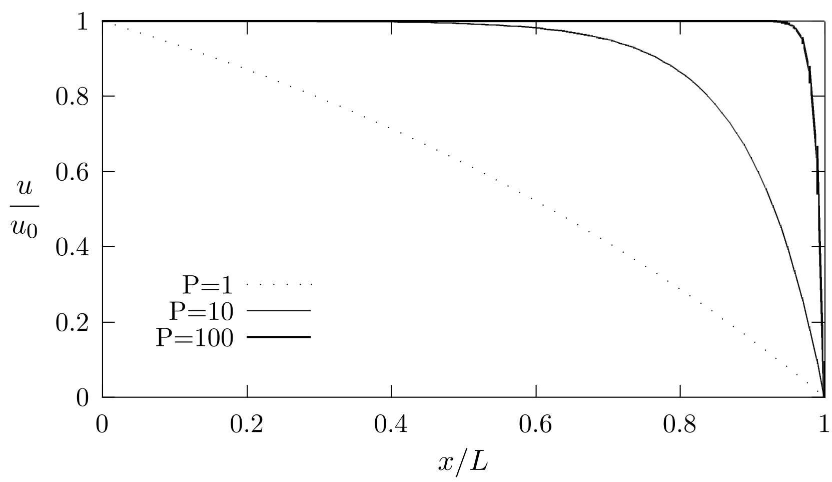
Thus $u(x) = A\exp(\rho cvx/L) + B$. Using the boundary conditions gives

$$u(0) = u_0 \quad \Rightarrow \quad A + B = u_0$$
$$u(L) = 0 \quad \Rightarrow \quad B = -A\exp(\rho cvL/k) = -A\exp(P),$$

$$A = \frac{u_0}{1 - \exp(P)}, \quad B = -u_0\frac{\exp(P)}{1 - \exp(P)}.$$

The solution is thus

$$u(x) = \frac{u_0}{1 - \exp(P)}\left(\exp(Px/L) - \exp(P)\right)$$



Solution for the case 2: Substituting the trial solution $u(x) = \exp(rx)$ into the differential equation gives

$$(-kr^2 + b)\exp(rx) = 0 \quad \Rightarrow \quad r = \pm\sqrt{b/k} = \pm\beta/L.$$

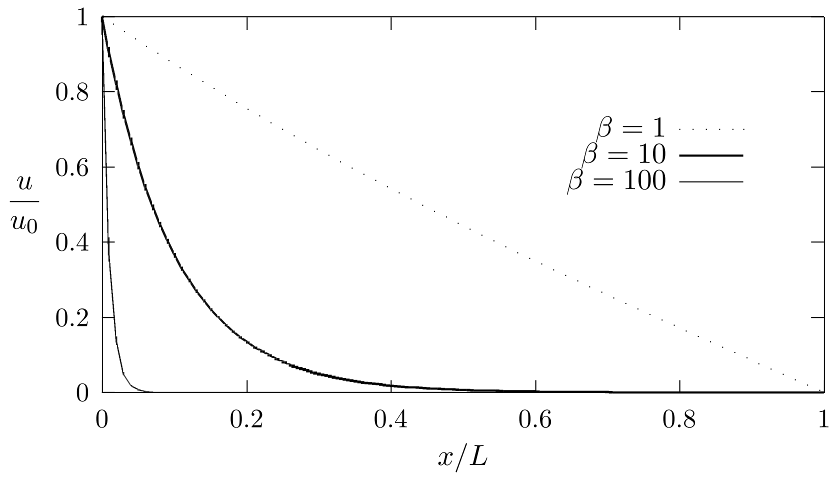
Thus $u(x) = A\exp(\beta x/L) + B\exp(-\beta x/L)$. Using the boundary conditions gives

$$\begin{aligned} u(0) = u_0 \quad &\Rightarrow \quad A + B = u_0 \\ u(L) = 0 \quad &\Rightarrow \quad B = -A\exp(2\beta) \end{aligned}$$

$$A = \frac{u_0}{1 - \exp(2\beta)}, \quad B = -u_0\frac{\exp(2\beta)}{1 - \exp(2\beta)}.$$

The solution is thus

$$u(x) = \frac{u_0}{1 - \exp(2\beta)}\left(\exp(\beta x/L) - \exp(2\beta)\exp(-\beta x/L)\right).$$



Problem 2

Adjoint operator D^* for a differential operator D in a domain Ω is defined with functions $u, v \in \mathcal{A}$ as

$$\int_\Omega vDud\Omega = \int_\Omega (D^*v)ud\Omega.$$

The operator D is self adjoint if $D = D^*$. Investigate which ones of the following operators are self adjoint:

$$D = -\frac{d^2}{dx^2}, \quad \mathcal{A} = \{u | u \in C_2(0, L), u(0) = u(L) = 0\} \tag{2a}$$

$$D = -\frac{d^2}{dx^2} + k\frac{d}{dx}, \quad \mathcal{A} = \{u | u \in C_2(0, L), u(0) = u(L) = 0\}. \tag{2b}$$

Notation $C_n(0, L)$ denotes a set of n-times continuously differentiable functions in an interval $(0, L)$, and k is a positive constant.

Solution for the case a: Integration by parts gives

$$-\int_0^L vu''dx = -\Big|_0^L vu' + \int_0^L v'u'dx = \int_0^L v'u'dx = \Big|_0^L v'u - \int_0^L v''u\,dx = -\int_0^L v''u\,dx$$

for all functions $u, v \in \mathcal{A}$, hence $D^* = -\frac{d^2}{dx^2}$ and $D^* = D$, thus the operator D is self adjoint.

Solution for the case b:

$$D = -\frac{d^2}{dx^2} + k\frac{d}{dx}, \quad \mathcal{A} = \{u | u \in C_2(0, L), u(0) = u(L) = 0\}$$

Solution:

$$\begin{aligned}
\int_0^L v(ku' - u'')dx &= \Big|_0^L (kvu - vu') - \int_0^L (kv'u - v'u')dx = \int_0^L (kv'u' - v'u)dx \\
&= \Big|_0^L v'u + \int_0^L (-v''u - kv'u)dx = -\int_0^L (v'' + kv')u\,dx \\
&\Rightarrow \quad D^* = -\frac{d^2}{dx^2} - k\frac{d}{dx} \neq D
\end{aligned}$$

D is not self adjoint.

Problem 3

A differential operator D is positive in a domain Ω with functions belonging to the set $\mathcal{A}$, if

$$\int_\Omega uDud\Omega > 0, \qquad \forall u \in \mathcal{A}.$$

Show, that the following operators are positive:

$$D_2 = -\frac{d}{dx}k\frac{d}{dx}, \quad \mathcal{A} = \{u | u \in C_2(0,L), u(0) = u(L) = 0\} \tag{3a}$$

$$D_4 = \frac{d^2}{dx^2}EI\frac{d^2}{dx^2}, \quad \mathcal{A} = \left\{u | u \in C_4(0,L), u(0) = u'(0) = u(L) = u'(L) = 0\right\}, \tag{3b}$$

and $k, EI > 0$. Notation $C_n(0,L)$ denotes a set of n-times continuously differentiable functions in an interval $\Omega = \{x | x \in (0,L)\}$.

Solution:

Let's investigate the integral $I(u)$:

$$I_2(u) = -\int_0^L u(ku')'dx = -\Big|_0^L uku' + \int_0^L k(u')^2dx = \int_0^L k(u')^2dx > 0$$

since if $u' \equiv 0$ then $u = c$=constant, and due to the boundary conditions $u(0) = u(L) = 0$ then u would vanish identically.

In a similar way for the fourth order operator

$$\begin{aligned} I_4(u) &= \int_0^L u(EIu'')''dx \\ &= \Big|_0^L u(EIu'')' - \int_0^L u'(EIu'')'dx \\ &= -\Big|_0^L u'EIu'' + \int_0^L EI(u'')^2dx \\ &= \int_0^L EI(u'')^2dx > 0, \end{aligned}$$

since if $u'' \equiv 0$ then $u = ax + b$, and since $u(0) = u(L) = 0$ then u would vanish identically.

Notice that the material/sectional constants have to be positive.

Problem 4

Investigate the type (parabolic/hyperbolic) of the following partial differential equations (PDEs)

$$\rho c \frac{\partial u}{\partial t} - \lambda \frac{\partial^2 u}{\partial x^2} = 0, \tag{4a}$$

$$\rho A \frac{\partial^2 u}{\partial t^2} - EA \frac{\partial^2 u}{\partial x^2} = 0, \tag{4b}$$

$$\rho A \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = 0, \tag{4c}$$

$$\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \times (\mu^{-1} \nabla \times \boldsymbol{A}) = \boldsymbol{0}. \tag{4d}$$

Are the hyperbolic equations dispersive?

Hint: The PDE is hyperbolic if substituting expression $u = \exp(i(\omega t + kx))$ into the equation gives real solutions for the frequency ω. The phase velocity is $c = \omega/k$. If the phase velocity depends on the wave number k, the problem is said to be dispersive.

Solution

Case a:

$$(\rho c \omega i + \lambda k^2) \exp(i(\omega t + kx)) = 0,$$

where i is the imaginary unit λ is the conductivity (usually denoted as k in this course, but now it could cause confusion). [1] It is immediately clear that the angular frequency ω has only imaginary solutions, thus the equation is parabolic.

Case b:

$$(-\rho A \omega^2 + EAk^2) \exp(i(\omega t + kx)) = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{E}{\rho}} k.$$

The phase velocity c is

$$c = \frac{\omega}{k} = \sqrt{\frac{E}{\rho}}$$

and it does not depend on the wave number k, the equation is not dispersive.

Case c:

$$(-\rho A \omega^2 + EIk^4) \exp(i(\omega t + kx)) = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{EI}{\rho A}} k^2.$$

The phase velocity c is

$$c = \frac{\omega}{k} = \sqrt{\frac{EI}{\rho A}} k.$$

Since the phase velocity depends on the wave number the problem **is dispersive**. In dispersive system waves of different wavelength travel with different velocities.

[1] In the literature the thermal conductivity has usually symbols like λ, k or κ.

The group velocity v_{R} is defined as

$$v_{\mathrm{R}} = \frac{d\omega}{dk} = 2\sqrt{\frac{EI}{\rho A}}k = 2c.$$

Case d:

$$\sigma\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \times (\mu^{-1}\nabla \times \boldsymbol{A}) = \boldsymbol{0}.$$

If the vector potential $\boldsymbol{A}$ has only one nonzero component A_z, the equation

$$\sigma\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \times (\mu^{-1}\nabla \times \boldsymbol{A}) = \boldsymbol{0}.$$

will reduce to (assuming also that the magnetic permeability μ is a constant)

$$\sigma\frac{\partial A_z}{\partial t} - \mu^{-1}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2}\right) = 0.$$

if the component A_z depend only on e.g. coordinate x, then the equation is as in the case (a). Thus the equation is parabolic.