MEI-55200 Numerical methods for field problems

1. Exercise: Mathematical preliminaries

1. For example, two problems where standard numerical schemes behave badly are in the stationary one-dimensional case: (a) the diffusion-convection equation and (b) the reaction-diffusion equation

$$-k\frac{d^2u}{dx^2} + \rho c v\frac{du}{dx} = 0, \tag{1a}$$

$$-k\frac{d^2u}{dx^2} + bu = 0, \quad \text{where} \quad b = \beta^2 k L^{-2} \tag{1b}$$

and β is a dimensionless parameter. It is assumed here that the physical parameters k, ρ, c, v, b are all constants in the domain $\Omega = \{x|x \in (0, L)\}$. Solve the problem with boundary conditions $u(0) = u_0 > 0, u(L) = 0$. Draw the solution with different values of the non-dimensional Péclet number $P = \rho c v L/k$, e.g. $P = 1, 10, 100$, and $\beta^2 = 1, 10, 100$. What happens when $P \to \infty$ and $\beta \to \infty$?

2. Adjoint operator D^* for a differential operator D in a domain Ω is defined with functions $u, v \in \mathcal{A}$ as

$$\int_\Omega vDud\Omega = \int_\Omega (D^*v)ud\Omega.$$

The operator D is self adjoint if $D = D^*$. Investigate which ones of the following operators are self adjoint:

$$D = -\frac{d^2}{dx^2}, \quad \mathcal{A} = \{u|u \in C_2(0, L), u(0) = u(L) = 0\} \tag{2a}$$

$$D = -\frac{d^2}{dx^2} + k\frac{d}{dx}, \quad \mathcal{A} = \{u|u \in C_2(0, L), u(0) = u(L) = 0\}\,. \tag{2b}$$

Notation $C_n(0, L)$ denotes a set of n-times continuously differentiable functions in an interval $(0, L)$, and k is a positive constant.

3. A differential operator D is positive in a domain Ω with functions belonging to the set $\mathcal{A}$, if

$$\int_\Omega uDud\Omega > 0, \quad \forall u \in \mathcal{A}.$$

Show, that the following operators are positive:

$$D_2 = -\frac{d}{dx}k\frac{d}{dx}, \quad \mathcal{A} = \{u|u \in C_2(0, L), u(0) = u(L) = 0\} \tag{3a}$$

$$D_4 = \frac{d^2}{dx^2}EI\frac{d^2}{dx^2}, \quad \mathcal{A} = \left\{u|u \in C_4(0, L), u(0) = u'(0) = u(L) = u'(L) = 0\right\}, \tag{3b}$$

and $k, EI > 0$. Notation $C_n(0, L)$ denotes a set of n-times continuously differentiable functions in an interval $\Omega = \{x|x \in (0, L)\}$.

4. Investigate the type (parabolic/hyperbolic) of the following partial differential equations (PDEs)

$$\rho c \frac{\partial u}{\partial t} - \lambda \frac{\partial^2 u}{\partial x^2} = 0, \tag{4a}$$

$$\rho A \frac{\partial^2 u}{\partial t^2} - EA \frac{\partial^2 u}{\partial x^2} = 0, \tag{4b}$$

$$\rho A \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = 0, \tag{4c}$$

$$\sigma \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \times (\mu^{-1} \nabla \times \boldsymbol{A}) = \boldsymbol{0}. \tag{4d}$$

In the case (d) you can assume that the vector potential $\boldsymbol{A}$ has only one nonzero component A_z which depends on one spatial coordinate.

Are the hyperbolic equations dispersive?

Hint: The PDE is hyperbolic if substituting expression $u = \exp(i(\omega t + kx))$ into the equation gives real solutions for the frequency ω. The phase velocity is $c = \omega/k$. If the phase velocity depends on the wave number k, the problem is said to be dispersive.

Home exercise: Let's investigate a pull-out-test of a reinforcement fibre. It can be modelled by equation

$$-E_{\mathrm{f}} A_{\mathrm{f}} \frac{d^2 u}{dx^2} + G_{\mathrm{m}} u = 0, \qquad u(0) = 0, \quad N(L) = F \tag{5}$$

where the normal force in the fibre is $N = E_{\mathrm{f}} A_{\mathrm{f}} u'$. The shear modulus of the matrix is assumed to be expressed in the form $G_{\mathrm{m}} = \beta^2 E_{\mathrm{f}} A_{\mathrm{f}} / L^2$, which gives $\beta^2 = G_{\mathrm{m}} L^2 / E_{\mathrm{f}} A_{\mathrm{f}}$.

Solve the problem analytically and draw the results, i.e the solution curves of the displacement u, the axial force N and the reaction force in the matrix $H = G_{\mathrm{m}} u$ in cases where $\beta = 1$ and 10.

What is the value of β for a $L = 10$cm long carbon fibre with a diameter 1 mm in an epoxy resin matrix?

To be returned at latest in the next exercise!