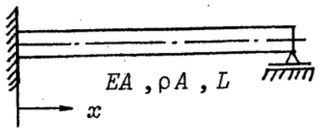
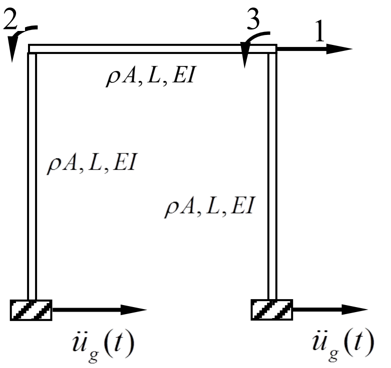


1. Määritä kuvan tasapaksun ja homogeenisen sauvan alimmat ominaiskulmataajuudet käyttämällä
   a) tarkkaa ratkaisua,
   b) yhtä sauvaelementtiä ja konsistenttia massamatriisia,
   c) yhtä sauvaelementtiä ja keskitettyä massamatriisia,
   d) kahta sauvaelementtiä ja konsistenttia massamatriisia,
   e) kahta sauvaelementtiä ja keskitettyä massamatriisia,



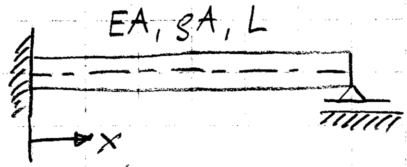
2. Alustaan vaikuttaa maanjäristyksessä tunnettu kiihtyvyysheräte $\ddot{u}_g(t)$. Käytä aikaisempien harjoitusten ratkaisua ja esitä järjestelmän liikeyhtälöt. Kerro miten laskisit kyseisen tehtävän? Voit myös ratkaista tehtävän vaikkapa vakioisella alustan kiihtyvyysherätteellä.

$$\mathbf{M} = \frac{\rho AL}{420}\begin{bmatrix} 732 & 22L & 22L \\ 22L & 8L^2 & -3L^2 \\ 22L & -3L^2 & 8L^2 \end{bmatrix}$$

$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}$$

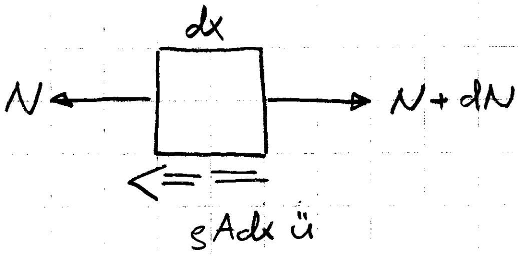
Rak. dyn.

1



sauvaelementti

a)



$$\sum F \rightarrow: \quad \cancel{-N} + \cancel{N} + dN - \rho A\, dx\, \ddot{u} = 0 \quad \| : dx$$

$$\frac{dN}{dx} - \rho A \ddot{u} = 0$$

$$\sigma = \frac{N}{A} = E\varepsilon \quad \Rightarrow \quad N = E\varepsilon A = EA\frac{du}{dx} \quad \Rightarrow \quad \frac{dN}{dx} = EA\frac{d^2u}{dx^2}$$

$L_0 = dx$

$L = dx + du$

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{dx + du - dx}{dx} = \frac{du}{dx}$$

<u>ODY:</u> $$\cancel{E}\cancel{A}\frac{d^2u}{dx^2} - \rho \cancel{A} \ddot{u} = 0$$

yrite: $\tilde{u} = \hat{u}\sin(\omega t)$

$\ddot{\tilde{u}} = -\omega^2 \hat{u} \sin(\omega t)$

$$\frac{E}{\rho}\frac{d^2\hat{u}}{dx^2}\sin(\omega t) + \omega^2\hat{u}\sin(\omega t) = 0$$

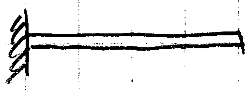
$$\frac{E}{\rho}\hat{u}'' + \omega^2\hat{u} = 0$$

$$\hat{u}'' + \frac{\omega^2\rho}{E}\hat{u} = 0$$

yrite. $$\hat{u} = C_1\cos\left(\omega\sqrt{\frac{\rho}{E}}\,x\right) + C_2\sin\left(\omega\sqrt{\frac{\rho}{E}}\,x\right)$$

$$\hat{u}' = -\omega\sqrt{\frac{\rho}{E}}\,C_1\sin\left(\omega\sqrt{\frac{\rho}{E}}\,x\right) + \omega\sqrt{\frac{\rho}{E}}\,C_2\cos\left(\omega\sqrt{\frac{\rho}{E}}\,x\right)$$

reunaehdot.



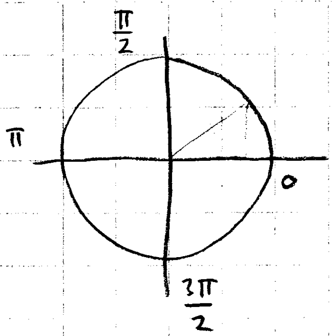
$$u(0) = 0 \quad \Rightarrow \quad C_1 = 0$$

$$u'(L) = \omega \sqrt{\frac{\rho}{E}}\, C_2 \cos\left(\omega \sqrt{\frac{\rho}{E}}\, L\right) = 0$$

$$\cos\left(\omega \sqrt{\frac{\rho}{E}}\, L\right) = 0 \quad \Rightarrow \quad \omega L \sqrt{\frac{\rho}{E}} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$= \left(\frac{\pi}{2} + n\pi\right) \qquad n = 0, 1, 2, \dots$$

$$= (2n+1)\frac{\pi}{2}$$



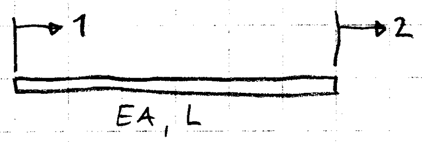
$$\Rightarrow \quad \omega = (2n+1)\frac{\pi}{2}\sqrt{\frac{E}{\rho L^2}}, \quad n = 0, 1, 2, 3 \dots$$

$$\omega_1 = 1{,}571 \sqrt{\frac{E}{\rho L^2}}$$

$$\omega_2 = 4{,}712 \sqrt{\frac{E}{\rho L^2}}$$

4/2

sauvaelementti

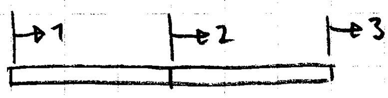


$$[K^e] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [m^e]_L = \rho AL \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad [m^e]_c = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

b) $$\frac{EA}{L} - \omega^2 \frac{\rho AL}{3} = 0 \quad \Rightarrow \quad \omega = \sqrt{3}\sqrt{\frac{E}{\rho L^2}} \approx 1{,}732\sqrt{\frac{E}{\rho L^2}} \quad (+10{,}2\%)$$

c) $$\frac{EA}{L} - \omega^2 \frac{\rho AL}{2} = 0 \quad \Rightarrow \quad \omega = \sqrt{2}\sqrt{\frac{E}{\rho L^2}} \approx 1{,}414\sqrt{\frac{E}{\rho L^2}} \quad (-7{,}9\%)$$

kaksi elementtiä



d) $$[K] = \frac{EA}{L}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \qquad [M_c] = \frac{\rho AL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

$$\det(K - \lambda M) = 0 \Rightarrow \begin{vmatrix} 4 - 4\tilde{\lambda} & -2 - \tilde{\lambda} \\ -2 - \tilde{\lambda} & 2 - 2\tilde{\lambda} \end{vmatrix} = 0 \qquad \tilde{\lambda} = \frac{\omega^2}{12}\frac{\rho AL^2}{EA}$$

$$\Rightarrow \quad 8 - 16\tilde{\lambda} + 8\tilde{\lambda}^2 - (4 + 4\tilde{\lambda} + \tilde{\lambda}^2) = 0$$

$$7\tilde{\lambda}^2 - 2\tilde{\lambda} + 4 = 0 \quad \Rightarrow \quad \tilde{\lambda}_1 = 0{,}2164$$

$$\tilde{\lambda}_2 = 2{,}6408$$

$$\Rightarrow \quad \omega_1 \approx 1{,}611\sqrt{\frac{E}{\rho L^2}} \quad (+2{,}5\%)$$

$$\omega_2 \approx 5{,}629\sqrt{\frac{E}{\rho L^2}} \quad (+19{,}5\%)$$

ominaisvektorit : $\phi_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1 \end{bmatrix}$

e) $[M]_L = \frac{\rho A L}{4}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 4-2\tilde{\lambda} & -2 \\ -2 & 2-\tilde{\lambda} \end{vmatrix} = 0 \qquad \tilde{\lambda} = \frac{\omega^2}{4}\frac{\rho A L^2}{EA}$$

$$\Rightarrow \quad 2\tilde{\lambda}^2 - 8\tilde{\lambda} + 4 = 0 \quad \Rightarrow \quad \tilde{\lambda}_1 = 0{,}586$$

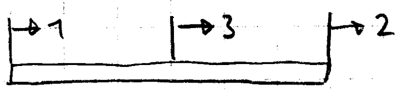
$$\tilde{\lambda}_2 = 3{,}414$$

$$\Rightarrow \quad \omega_1 \approx 1{,}531\sqrt{\frac{E}{\rho L^2}} \qquad (-2{,}5\%)$$

$$\omega_2 \approx 3{,}696\sqrt{\frac{E}{\rho L^2}} \qquad (-21{,}6\%)$$

$$\{\tilde{\phi}_1\} = \begin{bmatrix} 1/\sqrt{2} \\ 1 \end{bmatrix} \qquad \{\tilde{\phi}\}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1 \end{bmatrix}$$

f) kolmisolmuinen sauva



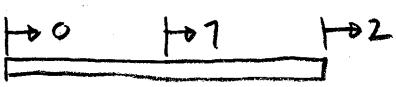
$$N_1(x) = 1 - 3\frac{x}{L} + 2\left(\frac{x}{L}\right)^2 \qquad N_{1,x} = \frac{1}{L}\left(-3 + 4\frac{x}{L}\right)$$

$$N_2(x) = -\frac{x}{L} + 2\left(\frac{x}{L}\right)^2 \qquad N_{2,x} = \frac{1}{L}\left(-1 + 4\frac{x}{L}\right)$$

$$N_3(x) = 4\frac{x}{L} - 4\left(\frac{x}{L}\right)^2 \qquad N_{3,x} = \frac{1}{L}\left(4 - 8\frac{x}{L}\right)$$

$$k_{ij} = \int_0^L EA\, N_{i,x} N_{j,x}\, dx \quad \Rightarrow \quad [k^e] = \frac{EA}{3L}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \end{array}\begin{array}{c} 0 \\ 1 \\ 2 \end{array}$$

$$m_{ij} = \int_0^L \rho A N_i N_j\, dx \quad \Rightarrow \quad [m^e] = \frac{\rho A L}{30}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix} \end{array}\begin{array}{c} 0 \\ 1 \\ 2 \end{array}$$



$$[K] = \frac{EA}{3L}\begin{bmatrix} 7 & -8 \\ -8 & 16 \end{bmatrix} \qquad [M] = \frac{\rho A L}{30}\begin{bmatrix} 4 & 2 \\ 2 & 16 \end{bmatrix}$$

①/4

$$\det(K - \omega^2 M) = 0$$

$$\Rightarrow \quad \omega_1 \approx 1{,}577 \sqrt{\frac{E}{\varrho L^2}} \qquad (+0{,}4\,\%)$$

$$\omega_2 \approx 5{,}673 \sqrt{\frac{E}{\varrho L^2}} \qquad (+20{,}4\,\%)$$

1/5

d) origon siirto valitaan $\varrho = 200 \frac{EI}{\varrho A L^4} = \bar{\varrho} \frac{EI}{\varrho A L^4}$

$$\hat{K} = K - \varrho M = \frac{EI}{L^3}\begin{bmatrix} -324{,}57 & -4{,}4762L & -4{,}4762L \\ -4{,}4762L & 4{,}1305L^2 & -3{,}42806L^2 \\ -4{,}4762L & 3{,}4286L^2 & 4{,}1305L^2 \end{bmatrix}$$

$\underline{\underline{L=1}}$

yrite

$$\tilde{\phi} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \tilde{y} = M\tilde{\phi} = \begin{pmatrix} 1{,}7429 \\ 0{,}026190 \\ 0{,}07857 \end{pmatrix} \qquad \bar{\lambda} = \lambda \frac{\varrho A L^4}{EI}$$

$$\tilde{v} = \hat{K}^{-1}\tilde{y} = \begin{pmatrix} -0{,}0054707 \\ -0{,}030714 \\ 0{,}0380597 \end{pmatrix} \qquad (K - \lambda M) = (\bar{K} - \bar{\lambda}\bar{M})$$

$$= [(\bar{K} - \bar{\varrho}\bar{M}) - \hat{\lambda}\bar{M}]$$

$$\tilde{w} = M\tilde{y} = \begin{pmatrix} -0{,}0091511 \\ -0{,}0011433 \\ 0{,}00065732 \end{pmatrix} \qquad \hat{\lambda} = \bar{\lambda} - \bar{\varrho}$$

$$\hat{\tilde{\lambda}} = \frac{\tilde{v}^T\tilde{y}}{\tilde{v}^T\tilde{w}} = -66{,}714 \qquad \tilde{y} = \frac{\tilde{w}}{\sqrt{v^T w}} = \begin{pmatrix} -0{,}87181 \\ -0{,}108918 \\ 0{,}06262 \end{pmatrix}$$

2. kierros

$$\tilde{v} = \hat{K}^{-1}\tilde{y} = \begin{pmatrix} 0{,}002756 \\ -0{,}114010 \\ 0{,}111136 \end{pmatrix} \qquad \tilde{w} = M\tilde{y} = \begin{pmatrix} 0{,}0045999 \\ -0{,}0028226 \\ 0{,}003074 \end{pmatrix}$$

$$\hat{\tilde{\lambda}} = \frac{\tilde{v}^T\tilde{y}}{\tilde{v}^T\tilde{w}} = 25{,}15$$

$$\bar{\lambda}_2 = \hat{\lambda} + \bar{\varrho} = 25{,}15 + 200 = 225{,}15$$

$$\lambda_2 = \bar{\lambda}_2 \frac{EI}{\varrho A L^4} = 225{,}15 \frac{EI}{\varrho A L^4} \qquad \lambda_2^{taulukko} = 229{,}09 \frac{EI}{\varrho A L^4}$$

$$\omega_2 = 14{,}90\sqrt{\frac{EI}{\varrho A L^4}} \qquad \phi_2^{taulukko} = \begin{pmatrix} 0 \\ 4{,}369/L \\ -4{,}369/L \end{pmatrix}$$

$$\phi_2 = \frac{\tilde{v}}{\sqrt{\tilde{v} M \tilde{v}}} = \begin{pmatrix} 0{,}1048 \\ -4{,}385/L \\ -4{,}274/L \end{pmatrix}$$