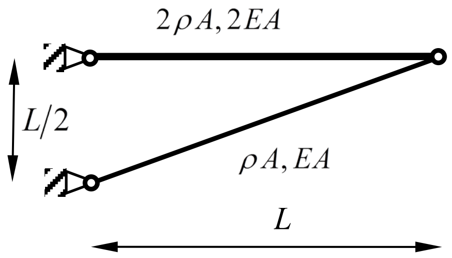
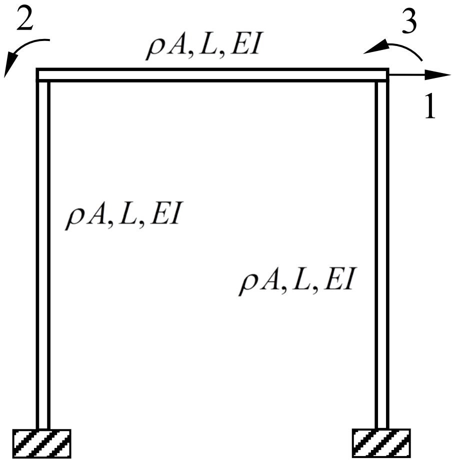
RAK-33046 Dynamics of Structures

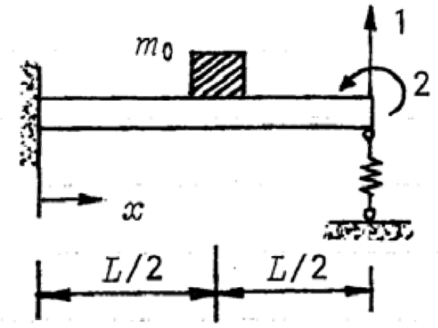
Exercise 2 22.1.2016



1. Determine the stiffness and consistent mass matrices of the truss. Calculate the eigenvalues and vectors of the system.



2. Determinate the consistent mass matrix of the enclosed frame using three DOFs.



3. The mass m_0 is added to the half of the beam. Determinate the consistent mass of the 2DOF system.

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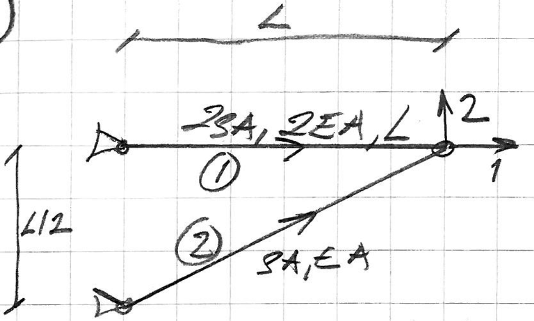


$$L_2 = \sqrt{L^2/4 + L^2} = \frac{\sqrt{5}}{2} L$$

$l = 1 \quad m = 0$

$$k^{①} = \frac{2EA}{L} \begin{array}{c} \\ \left[\begin{array}{cccc} & & 1 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \\ 1 \\ 2 \end{array}$$

$l = \frac{2}{\sqrt{5}} \quad m = \frac{1}{\sqrt{5}}$

$$k^{②} = \frac{EA \cdot 2}{\sqrt{5} L} \frac{1}{5} \begin{array}{c} \\ \left[\begin{array}{cccc} & & 1 & 2 \\ 4 & 2 & -4 & -2 \\ 2 & 1 & -2 & -1 \\ -4 & -2 & 4 & 2 \\ -2 & -1 & 2 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \\ 1 \\ 2 \end{array}$$

$$K = \frac{EA}{L} \begin{bmatrix} 2 + 8/(5\sqrt{5}) & 0 + 4/(5\sqrt{5}) \\ 4/(5\sqrt{5}) & 2/(5\sqrt{5}) \end{bmatrix} \approx \frac{EA}{L} \begin{bmatrix} 2{,}7155 & 0{,}3578 \\ 0{,}3578 & 0{,}1789 \end{bmatrix}$$

$$m^{①} = \frac{2\rho A L}{6} \begin{array}{cc} 1 & 2 \\ \left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right. & \left.\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \qquad m^{②} = \frac{\rho A \sqrt{5} L}{12} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$M = \rho A L \cdot \begin{bmatrix} 2/3 + \sqrt{5}/6 & 0 \\ 0 & 2/3 + \sqrt{5}/6 \end{bmatrix} \approx \rho A L \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1{,}0393 & 0 \\ 0 & 1{,}0393 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$\omega = \bar{\omega}\sqrt{E/\rho L^2} \qquad \sqrt{\frac{[k]}{[m]}} = \sqrt{\frac{EA}{\rho A L^2}} = \sqrt{\frac{E}{\rho L^2}}$$

$$\det(K-\omega^2 M) = \det\left(\begin{bmatrix} 2{,}715 & 0{,}3578 \\ 0{,}3578 & 0{,}1789 \end{bmatrix} - \bar{\omega}^2 \begin{bmatrix} 1{,}0393 & 0 \\ 0 & 1{,}0393 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 2{,}715-1{,}0393\bar{\omega}^2 & 0{,}3578 \\ 0{,}3578 & 0{,}1789-1{,}0393\bar{\omega}^2 \end{bmatrix}\right) = 0$$

$$1{,}0802\,\bar{\omega}^4 - 3{,}0083\,\bar{\omega}^2 - 0{,}3578 = 0$$

Roots: $\bar{\omega}_1^2 = 0{,}12443 \qquad \bar{\omega}_2^2 = 2{,}660$

$$\omega_1 = 0{,}35278\sqrt{E/\rho L^2}$$

$$\omega_2 = 1{,}631\sqrt{E/\rho L^2}$$

Eigenvectors for ω_1 $(\phi_1^1 = 1)$

$$\begin{bmatrix} 2{,}715-1{,}0393\cdot 0{,}1244 & 0{,}3578 \\ 0{,}3578 & 0{,}1789-1{,}0393\cdot 0{,}1244 \end{bmatrix}\begin{pmatrix} 1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\phi_2 = -7{,}22 \qquad \Rightarrow \underline{\phi}^1 = \begin{pmatrix} 1 \\ -7{,}22 \end{pmatrix} \quad \omega_1 = 0{,}35278\sqrt{E/\rho L^2}$

Eigenvectors for ω_2 $\quad \phi_1^2 = 1$

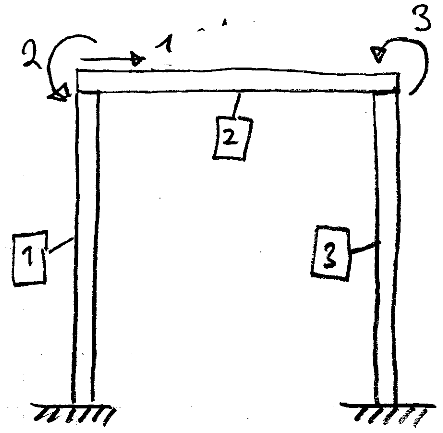
$$\begin{bmatrix} 2{,}715-1{,}0393\cdot 2{,}660 & 0{,}3578 \\ 0{,}3578 & 0{,}1789-1{,}0393\cdot 2{,}660 \end{bmatrix}\begin{pmatrix} 1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\phi_2 = +0{,}138 \qquad \underline{\phi}^2 = \begin{pmatrix} 1 \\ 0{,}138 \end{pmatrix} \quad \omega_2 = 1{,}631\sqrt{E/\rho L^2}$

$\underline{\phi}^1 = \begin{pmatrix} 1 \\ -7{,}22 \end{pmatrix} \sim \begin{pmatrix} 0{,}138 \\ -1 \end{pmatrix}$

$\underline{\phi}^2 = \begin{pmatrix} 1 \\ 0{,}138 \end{pmatrix}$

$\underline{\phi}^1 \perp \underline{\phi}^2 \quad , \quad \underline{\phi}^1 \underset{K,M}{\perp} \underline{\phi}^2$



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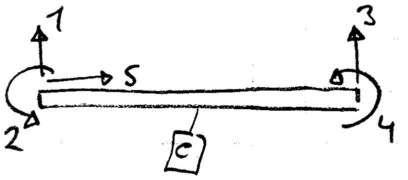

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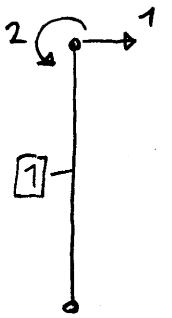
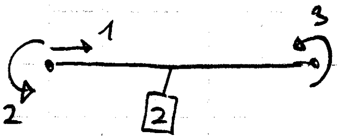
$$m^e_c = \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L & 0 \\ & 4L^2 & 13L & -3L^2 & 0 \\ & & 156 & -22L & 0 \\ & \text{symm.} & & 4L^2 & 0 \\ & & & & 420 \end{bmatrix}$$

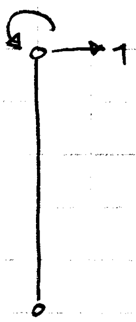
$$k^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 \\ & 4L^2 & -6L & 2L^2 & 0 \\ & & 12 & -6L & 0 \\ & \text{symm.} & & 4L^2 & 0 \\ & & & & 0 \end{bmatrix}$$



$$m^1_c = \frac{\rho A L}{420}\begin{array}{c} \\ \end{array}\begin{array}{cc|c} 1 & 2 & \\ \hline 156 & 22L & 1 \\ 22L & 4L^2 & 2 \end{array} , \quad k^1 = \frac{EI}{L^3}\begin{array}{cc|c} 1 & 2 & \\ \hline 12 & 6L & 1 \\ 6L & 4L^2 & 2 \end{array}$$



$$m^2_c = \frac{\rho A L}{420}\begin{array}{ccc|c} 1 & 2 & 3 & \\ \hline 420 & 0 & 0 & 1 \\ 0 & 4L^2 & -3L^2 & 2 \\ 0 & -3L^2 & 4L^2 & 3 \end{array} , \quad k^2 = \frac{EI}{L^3}\begin{array}{ccc} 1 & 2 & 3 \\ \hline 0 & 0 & 0 \\ 0 & 4L^2 & 2L^2 \\ 0 & 2L^2 & 4L^2 \end{array}$$



$$m^3_c = \frac{\rho A L}{420}\begin{array}{cc|c} 1 & 3 & \\ \hline 156 & 22L & 1 \\ 22L & 4L^2 & 3 \end{array} , \quad k^3 = \frac{EI}{L^3}\begin{array}{cc|c} 1 & 3 & \\ \hline 12 & 6L & 1 \\ 6L & 4L^2 & 3 \end{array}$$

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$$M_c = \frac{\rho A L}{420}\begin{bmatrix} 156+420+156 & 22L+0 & 22L \\ 22L+0 & 4L^2+4L^2 & -3L^2 \\ 22L & -3L^2 & 4L^2+4L^2 \end{bmatrix}$$

$$= \frac{\rho A L}{420}\begin{bmatrix} 732 & 22L & 22L \\ 22L & 8L^2 & -3L^2 \\ 22L & -3L^2 & 8L^2 \end{bmatrix}$$

$$K = \frac{EI}{L^3}\begin{bmatrix} 12+0+12 & 6L+0 & 6L \\ 6L+0 & 4L^2+4L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2+4L^2 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}$$

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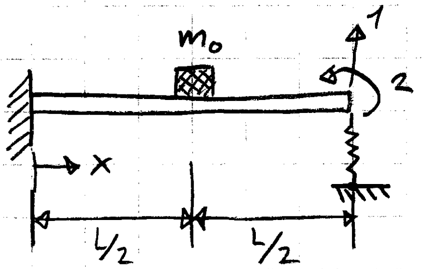
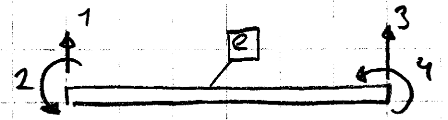
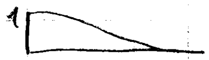


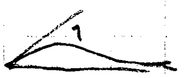


HERMITEN Polynomit

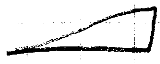
$$N_1^e = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}$$



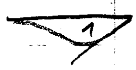
$$N_2^e = L\left(\frac{x}{L} - 2\frac{x^2}{L^2} + \frac{x^3}{L^3}\right)$$



$$N_3^e = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}$$



$$N_4^e = L\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right)$$



$$M = \rho A \int_0^L N^T N \, dx + m_0 N^T N$$

$$N = [\, N_1 \quad N_2 \,] \qquad N^T N = \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix}$$

$$\Rightarrow M_{ij} = \rho A \int_0^L N_i N_j \, dx + m_0 N_i N_j$$

$$M_{11} = \rho A \int_0^L N_3^{e2} \, dx + m_0 \left(N_1\left(\tfrac{L}{2}\right)\right)^2$$

$$= \rho A \int_0^L \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)^2 dx + m_0\left(\frac{3}{4}\frac{L^2}{L^2} - \frac{2}{8}\frac{L^3}{L^3}\right)^2$$

$$= \rho A \int_0^L \left(9\frac{x^4}{L^4} - 12\frac{x^5}{L^5} + 4\frac{x^6}{L^6}\right) dx + m_0\left(\frac{1}{2}\right)^2$$

$$= \rho A \Big/_0^L \left(\frac{9}{5}\frac{x^5}{L^4} - \frac{12}{6}\frac{x^6}{L^5} + \frac{4}{7}\frac{x^7}{L^6}\right) dx + \frac{1}{4} m_0$$

$$= \rho A \left(\frac{9}{5} - 2 + \frac{4}{7}\right) L + \frac{1}{4} m_0 = \frac{13}{35}\rho A L + \frac{1}{4} m_0$$

Liike-energia:

$$T_{m_0} = \frac{1}{2} m_0 v^2 = \frac{1}{2}\dot{q}^T N^T N \dot{q}\, m_0 = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$v\left(x = \frac{L}{2}\right) = \underline{N}\,\underline{\dot{q}} \qquad M = m_0 N^T N$$

$$N\left(x = \frac{L}{2}\right)$$

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$$M_{12} = \rho A \int_0^L N_3^e N_4^e \, dx \quad + \quad m_0 \, N_3^e(\tfrac{L}{2}) \cdot N_4^e(\tfrac{L}{2})$$

$$= \rho A \int_0^L \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right)\left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right) L \, dx \; + \; m_0 \left(3 \cdot \frac{1}{4} - 2 \cdot \frac{1}{8}\right)\left(-\frac{1}{4} + \frac{1}{8}\right) L$$

$$= \rho A \int_0^L \left(-3\frac{x^4}{L^4} + 3\frac{x^5}{L^5} + 2\frac{x^5}{L^5} - 2\frac{x^6}{L^6}\right) L \, dx + \left(-\frac{1}{16}\right) m_0 L$$

$$= \rho A \Big/_0^L \left(-\frac{3}{5}\frac{x^5}{L^4} + \frac{5}{6}\frac{x^6}{L^5} - \frac{2}{7}\frac{x^7}{L^6}\right) L \; - \; \frac{1}{16} m_0 L$$

$$= -\frac{11}{210} \rho A L^2 \; - \; \frac{1}{16} m_0 L$$

$$M_{22} = \rho A \int_0^L (N_4^e)^2 \, dx + m_0 \left(N_4^e(\tfrac{L}{2})\right)^2$$

$$= \rho A \int_0^L L^2 \left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right)^2 dx \; + \; m_0 L^2 \left(-\frac{1}{4}\frac{L^2}{L^2} + \frac{1}{8}\frac{L^3}{L^3}\right)^2$$

$$= \rho A L^2 \int_0^L \left(\frac{x^4}{L^4} - 2\frac{x^5}{L^5} + \frac{x^6}{L^6}\right) dx \; + \; m_0 L^2 \left(-\frac{1}{8}\right)^2$$

$$= \rho A L^2 \Big/_0^L \left(\frac{1}{5}\frac{x^5}{L^4} - \frac{2}{6}\frac{x^6}{L^5} + \frac{1}{7}\frac{x^7}{L^6}\right) + \frac{1}{64} m_0 L^2$$

$$= \frac{1}{105} \rho A L^3 \; + \; \frac{1}{64} m_0 L^2$$

$$\Rightarrow [M] = \frac{\rho A L}{420} \begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix} + \frac{m_0}{64} \begin{bmatrix} 16 & -4L \\ -4L & L^2 \end{bmatrix}$$

$$\Rightarrow [M] \; = \; [m]_c \; + \; \frac{m_0}{64} \begin{bmatrix} 16 & -4L \\ -4L & L^2 \end{bmatrix}$$

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