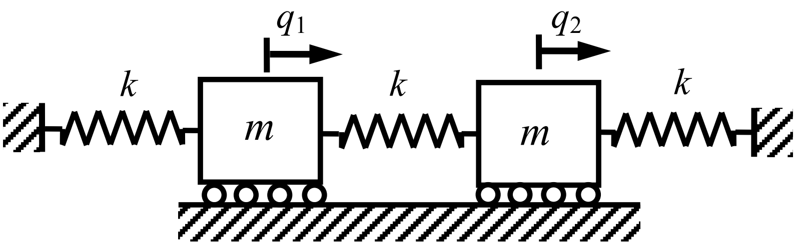
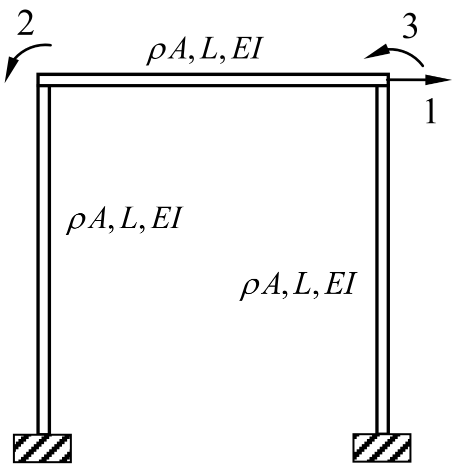
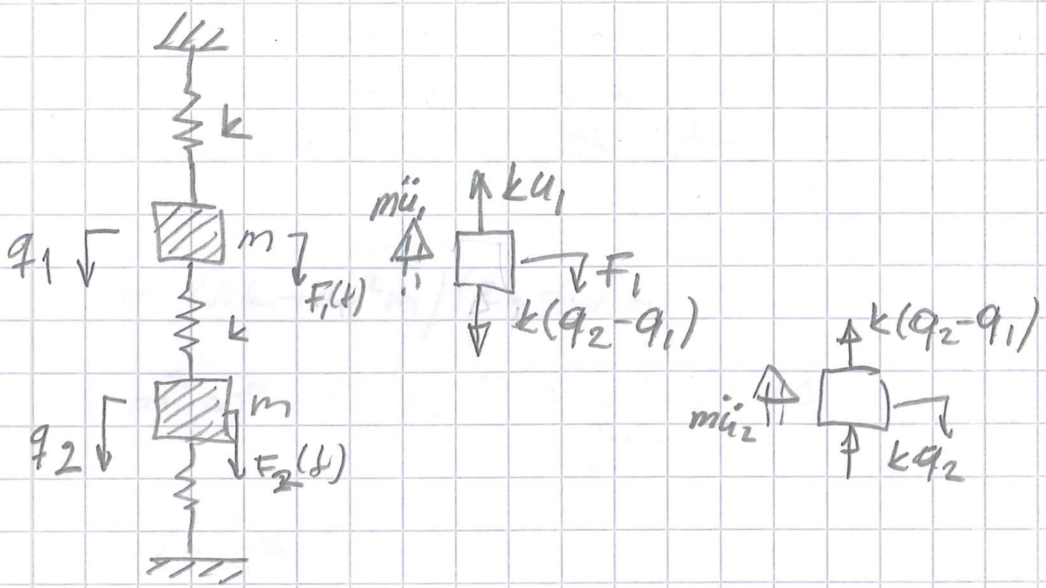


1. Consider above mass spring system. Determinate the equation of motion and calculate the eigenvalues and eigenvectors of the system. Draw the eigenvectors.



2. Determinate the stiffness matrix of the enclosed frame using three DOFs. The axial stiffness can be neglected. Exploit the lumped mass matrix approximation and determinate the non-zero eigenvalue and eigenvector.

   Derive single DOF system by statically condensing the rotational DOFs leaving the lateral DOF.



$\Sigma F \uparrow$ (1) $\qquad m\ddot q_1 + kq_1 - k(q_2-q_1) - F_1 = 0$

$\Sigma F \uparrow$ (2) $\qquad m\ddot q_2 + k(q_2-q_1) + kq_2 - F_2 = 0$

$$\underset{M}{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}} \begin{pmatrix} \ddot q_1 \\ \ddot q_2 \end{pmatrix} + \underset{K}{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \underset{F}{\cancel{\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}}}\ 0$$

$\underline{q} = \underline{\phi} \sin\omega t \qquad \dot{\underline{q}} = \omega \underline{\phi} \cos\omega t \qquad \ddot{\underline{q}} = -\omega^2 \underline{\phi} \sin\omega t$

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \sin\omega t + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \sin\omega t = \underline{0}$$

$$\underbrace{\left(\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right)}_{\det = 0} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \sin\omega t = \underline{0}$$

Ei triviaali ratkaisu

$\det(K - \omega^2 M) = 0$

$$\begin{vmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{vmatrix}$$

$$= (2k-\omega^2 m)(2k-\omega^2 m) - k^2 = 0$$

$$m^2\omega^4 - 4mk\omega^2 + \underbrace{4k^2 - k^2}_{3k^2} = 0$$

$$\lambda = \omega^2$$

$$m^2\lambda^2 - 4mk\lambda + 3k^2 = 0$$

$$\lambda_{1,2} = \frac{+4mk \pm \sqrt{16m^2k^2 - 12m^2k^2}}{2m^2}$$

$$= \frac{+4mk \pm 2mk}{2m} = \frac{2k \pm k}{m}$$

$\lambda_1 = k/m \qquad \lambda_2 = 3k/m$

$[k] = N/m = kg/s^2$

$[m] = kg$

$\omega_1 = \sqrt{k/m} \qquad \omega_2 = \sqrt{3k/m}$

$[\omega_1] = \sqrt{kg/s^2/kg}$

$= 1/s$

$$f = \frac{\omega}{2\pi}$$

$\omega = \omega_1 = \sqrt{k/m}$

MATLAB $[\phi, \lambda] = eig(K, M)$

$$\Rightarrow \begin{bmatrix} 2k-k & -k \\ -k & 2k-k \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

val $\phi_1 = 1 \Rightarrow \phi_2 = 1$

$$\underline{\phi}^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

L1 (3)

$\omega = \omega_2 = \sqrt{3k/m}$

$$\begin{bmatrix} 2k-3k & -k \\ -k & 2k-3k \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{val } \phi_1 = 1$$

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

val $\phi_1 = 1 \Rightarrow \phi_2 = -1$

$$\underline{\phi}^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

ominaisparit

$$\omega_1 = \sqrt{k/m} \quad \underline{\phi}^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2 = \sqrt{3k/m} \quad \underline{\phi}^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{\Phi} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} k/m & 0 \\ 0 & 3k/m \end{bmatrix}$$

$\underline{\phi}^1$ in $\underline{\phi}^2$ valdeen skeckta silen atte

$$\underline{\phi}^{iT} M \underline{\phi}^i = 1$$

$$\underline{\phi}^{1T} M \underline{\phi}^1 = 2m \qquad \underline{\phi}^{2T} M \underline{\phi}^2 = 2m$$

$$\hat{\underline{\phi}}^1 = \frac{1}{\sqrt{2m}} \underline{\phi}^1 = \begin{pmatrix} 1/\sqrt{2m} \\ 1/\sqrt{2m} \end{pmatrix} \qquad \hat{\underline{\phi}}^2 = \begin{pmatrix} 1/\sqrt{2m} \\ -1/\sqrt{2m} \end{pmatrix}$$

$\hat{\underline{\phi}}^1$

(4)

$$\underline{\emptyset} = \frac{1}{\sqrt{2m}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\underline{q} = \underline{\emptyset}\,\underline{\eta} \Rightarrow \ddot{\underline{q}} = \underline{\emptyset}\,\ddot{\underline{\eta}}$$

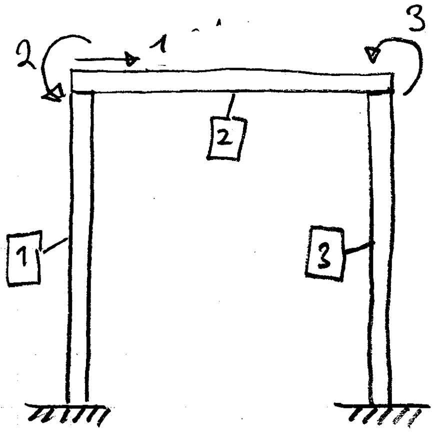
$$\emptyset^T \Big| \quad M\ddot{\underline{q}} + k\underline{q} = \underline{F}$$

$$\underline{\emptyset}^T M \underline{\emptyset}\,\ddot{\eta} + \emptyset^T K \underline{\emptyset}\,\underline{\eta} = \underline{\emptyset}^T \underline{F}$$

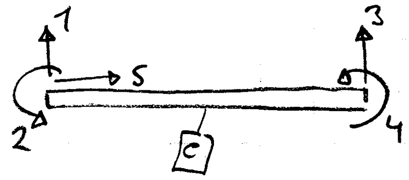
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{pmatrix} + \begin{bmatrix} k/m & 0 \\ 0 & 3k/m \end{bmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \underline{\emptyset}^T \underline{F}$$

$$\underbrace{\frac{1}{\sqrt{2m}^2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} m & m \\ m & -m \end{bmatrix}}}_{\begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}}$$

3

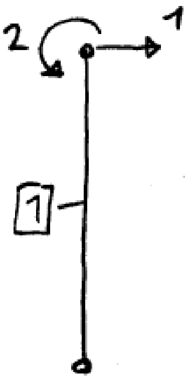


EB-palkki

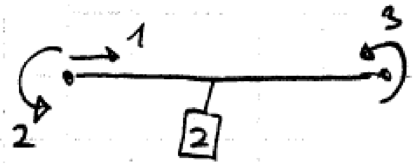


EB-palkielementin matriisit

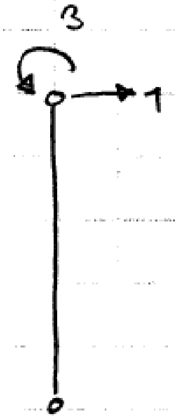
$$k^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 \\ & 4L^2 & -6L & 2L^2 & 0 \\ & & 12 & -6L & 0 \\ \text{symm.} & & & 4L^2 & 0 \\ & & & & 0 \end{bmatrix}$$



$$k^1 = \frac{EI}{L^3}\begin{array}{c c} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$



$$k^2 = \frac{EI}{L^3}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4L^2 & 2L^2 \\ 0 & 2L^2 & 4L^2 \end{bmatrix} \end{array}$$



$$k^3 = \frac{EI}{L^3} \begin{array}{cc} \begin{array}{cc} 1 & 3 \end{array} & \\ \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} & \begin{array}{c} 1 \\ 3 \end{array} \end{array}$$

$$K = \frac{EI}{L^3} \begin{bmatrix} 12+0+12 & 6L+0 & 6L \\ 6L+0 & 4L^2+4L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2+4L^2 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}$$

$$[M] = \varrho A L \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad [K] = \frac{EI}{L^3} \begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}$$

$$\det([K] - \lambda [M]) = 0 \qquad \lambda = \frac{\varrho A L^4}{EI} \omega^2$$

$$\begin{vmatrix} 24-2\lambda & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{vmatrix} = 0$$

$$(24-2\lambda)60L^4 - 6L \cdot 36L^3 + 6L \cdot (-36L^3) = 0$$

$$1440\cancel{L^4} - 120\cancel{L^4}\lambda - 216\cancel{L^4} - 216\cancel{L^4} = 0$$

$$\lambda = 8{,}4$$

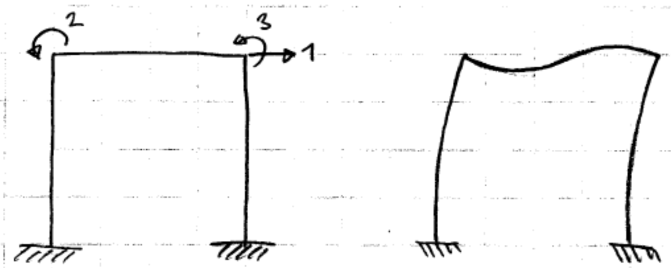
$$\Rightarrow \quad \omega = \sqrt{8{,}4} \sqrt{\frac{EI}{\varrho A L^4}} \approx 2{,}898$$

$$\begin{bmatrix} 8L^2 & 2L^2 \\ 2L^2 & 8L^2 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -6L \\ -6L \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -6/L \\ -6/L \end{bmatrix}$$

$$\begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}^{-1} \begin{bmatrix} -6/L \\ -6/L \end{bmatrix} = \frac{1}{\underbrace{64-4}_{60}} \begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} -6/L \\ -6/L \end{bmatrix}$$

$$\begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \frac{1}{60} \begin{bmatrix} -36/L \\ -36/L \end{bmatrix} = -\frac{3}{5L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0{,}6/L \\ -0{,}6/L \end{bmatrix}$$

$$\underline{\phi} = \begin{bmatrix} 1 \\ -0{,}6/L \\ -0{,}6/L \end{bmatrix}$$



$$\hat{\underline{\phi}}_1 = \frac{1}{\sqrt{\underline{\phi}^T M \underline{\phi}^T}} \underline{\phi} = \frac{1}{\sqrt{2\varrho AL}} \begin{bmatrix} 1 \\ -0{,}6/L \\ -0{,}6/L \end{bmatrix}$$

$\varrho AL = 18kg$

Static condencation

$$K = \frac{EI}{L^3}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{c|cc} 24 & 6L & 6L \\ \hline 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right]\frac{EI}{L^3}$$

$$K^* = K_{11} - K_{12}K_{22}^{-1}K_{21}$$

$$K_{22}^{-1} = \frac{1}{8\cdot 8L^4 - 2\cdot 2L^4}\begin{bmatrix} 8L^2 & -2L^2 \\ -2L^2 & 8L^2 \end{bmatrix} = \frac{1}{60L^2}\begin{bmatrix} 8L^2 & -2L^2 \\ -2L^2 & 8L^2 \end{bmatrix}$$

$$K_{12}\cdot K_{22}^{-1}K_{21} = \frac{EI}{L^3}\begin{bmatrix} 6L & 6L \end{bmatrix}\frac{1}{60L^4}\begin{bmatrix} 8L^2 & -2L^2 \\ -2L^2 & 8L^2 \end{bmatrix}\begin{bmatrix} 6L \\ 6L \end{bmatrix}$$

$$= \frac{EI}{60L^7}\begin{bmatrix} 6L & 6L \end{bmatrix}\begin{bmatrix} 36L^3 \\ 36L^3 \end{bmatrix} = \frac{432EI}{60L^3}$$

$$= 7{,}2EI/L^3$$

$$K^* = \frac{EI}{L^3}(24 - 7{,}2) = 16{,}8\frac{EI}{L^3}$$

$$M^* = 2\rho AL$$

$$(K^* - \omega^2 M^*)\phi = 0 \iff \left(16{,}8\frac{EI}{L^3} - \omega^2\cdot 2\rho AL\right)\phi = 0$$

$$\omega^2 = 8{,}4\frac{EI}{\rho AL^4}$$

$$\omega = \sqrt{8{,}4}\sqrt{\frac{EI}{\rho AL^4}} = 2{,}898\sqrt{\frac{EI}{\rho AL^4}}$$