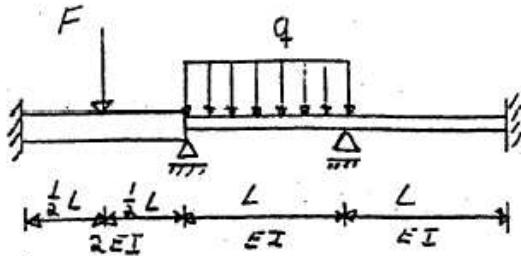


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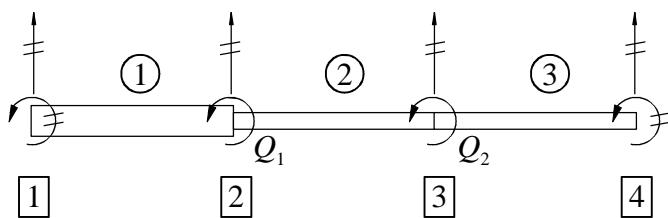
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Exercise 7 2013 A



1. Use three elements to solve the rotations of the middle supports. Draw also the bending moment diagram. $q = \frac{2F}{L}$.

We divide the structure to three beam elements with two nodal degrees of freedom. The nodes, elements and global degrees of freedom are drawn in the figure below



The active degrees of freedom are presented by the ID-table below

Elem	Node1	Node2
1	1	2
2	2	3
3	3	4

Node	dof1	dof2
1	0	0
2	0	1
3	0	2
4	0	0

The element stiffness matrices are formed next.

$$\mathbf{k}_1 = \frac{2EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} 0 \quad \mathbf{k}_2 = \frac{EI}{L^3} \begin{bmatrix} 0 & 1 & 0 & 2 \\ 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} 1 \quad \mathbf{k}_3 = \frac{EI}{L^3} \begin{bmatrix} 0 & 2 & 0 & 0 \\ 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} 2$$

The model stiffness matrix is scattered with the element stiffness matrix components.

$$\mathbf{K} = \sum_{i=1}^3 \mathbf{k}_i = \frac{EI}{L} \begin{bmatrix} 12 & 2 \\ 2 & 8 \end{bmatrix}$$

Element 1 external equivalent load

$$\mathbf{f}^p = \begin{bmatrix} -F/2 \\ -FL/8 \\ -F/2 \\ +FL/8 \end{bmatrix} 0$$

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Element 2 external equivalent load (q is negative)

$$\mathbf{f}^p = \begin{bmatrix} qL/2 \\ qL^2/12 \\ qL/2 \\ -qL^2/12 \end{bmatrix} = \begin{bmatrix} -F \\ -FL/6 \\ -F \\ +FL/6 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 0 \\ 2 \end{matrix}$$

The element external equivalent loads are collected to the table below

Ele	\mathbf{f}_1	\mathbf{f}_2	\mathbf{f}_3	\mathbf{f}_4
1	-F/2	-FL/8	-F/2	FL/8
2	-F	-FL/6	-F	FL/6
3	0	0	0	0

The global load vector is assembled from the element external loads

$$\mathbf{F} = \begin{bmatrix} FL/8 - FL/6 \\ FL/6 \end{bmatrix} = \frac{FL}{24} \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Now we may solve the global displacements

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \frac{FL^2}{24EI} \frac{1}{92} \begin{bmatrix} 8 & -2 \\ -2 & 12 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \frac{FL^2}{1104EI} \begin{bmatrix} -8 \\ 25 \end{bmatrix}$$

The element force vectors are calculated next

$$\mathbf{f}_1 = \mathbf{k}_1 \mathbf{q}_1 - \mathbf{f}^p = \frac{2EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \frac{FL^2}{1104EI} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -8 \end{bmatrix} + \begin{bmatrix} F/2 \\ FL/8 \\ F/2 \\ -FL/8 \end{bmatrix} = \frac{F}{552} \begin{bmatrix} 228 \\ 53L \\ 324 \\ -101L \end{bmatrix}$$

$$\mathbf{f}_2 = \mathbf{k}_2 \mathbf{q}_2 - \mathbf{f}^p = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \frac{FL^2}{1104EI} \begin{bmatrix} 0 \\ -8 \\ 0 \\ 25 \end{bmatrix} + \begin{bmatrix} F \\ FL/6 \\ F \\ -FL/6 \end{bmatrix} = \frac{F}{552} \begin{bmatrix} 603 \\ 101L \\ 501 \\ -50L \end{bmatrix}$$

$$\mathbf{f}_3 = \mathbf{k}_3 \mathbf{q}_3 = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \frac{FL^2}{1104EI} \begin{bmatrix} 25 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{F}{552} \begin{bmatrix} 75 \\ 50L \\ -75 \\ 25L \end{bmatrix}$$

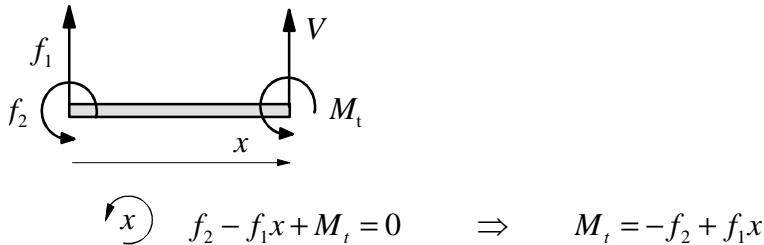
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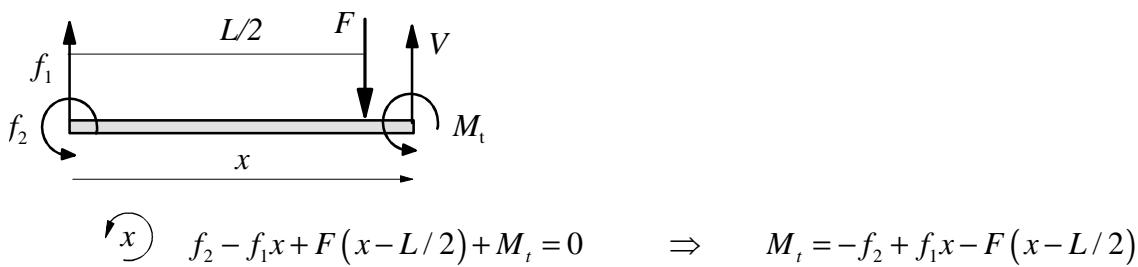
We check the vertical equilibrium by $\sum F_y = \frac{F}{552}(228+927+576-75) - 3F = 0$ OK

and the moment sum $M_{right} = -\frac{FL}{552}(228 \cdot 3 + 927 \cdot 2 + 576 \cdot 1 - 25 - 53) + \frac{5FL}{2} + \frac{6FL}{2} = 0$ OK

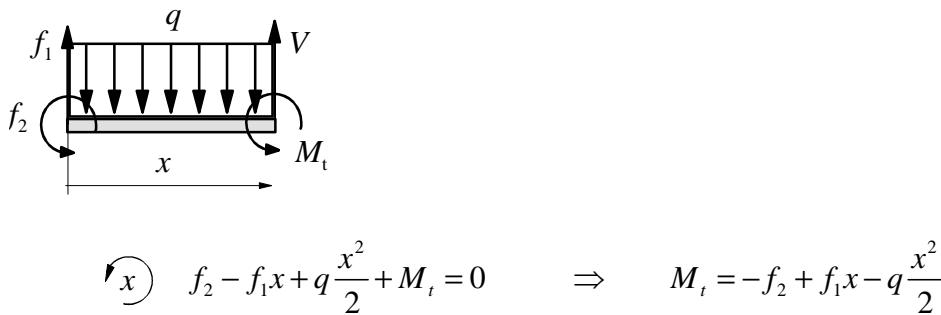
To draw a bending moment curve we need to consider three different cases in our model. At first we start from left edge towards to the point load.



Next we consider the position x after the point load F



Finally we consider the case where a force density acts on the element edge



The extremum value of M_t is obtained by setting the variation of the moment M_t to zero (tai derivoi ihan normaalisti x :n suhteen ja aseta lauseke nollaksi)

$$\delta M_t = f_1 \delta x - q x \delta x = 0 \quad \Rightarrow \quad x = \frac{f_1}{q}$$

By substituting the values

$$f_1 = \frac{603}{552} F \quad \text{and} \quad q = \frac{2F}{L}$$

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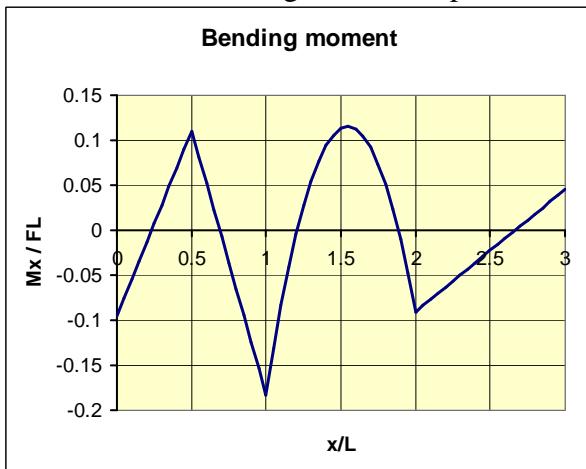
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we find that $x = \frac{603}{1104}L$ will produce the (local) extremum value of the moment M_t .

Now we are ready to draw the bending moment curve. We will still collect the results above to the table

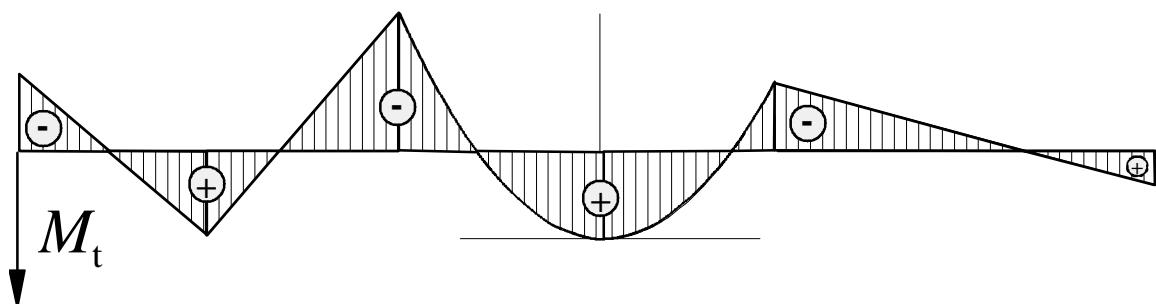
Distance from left	M_t
$z \leq \frac{L}{2}$	$M_t = -\frac{53FL}{552} + \frac{228}{552}Fx$
$\frac{L}{2} \leq z < L$	$M_t = -\frac{53FL}{552} + \frac{228}{552}Fx - F\left(x - \frac{L}{2}\right)$
$L \leq z < 2L$	$M_t = -\frac{101}{552}FL + \frac{603}{552}Fx - \frac{F}{L}x^2$
$2L \leq z \leq 3L$	$M_t = -\frac{50}{552}FL + \frac{75}{552}Fx$

The calculated bending moment is presented in the figure below.



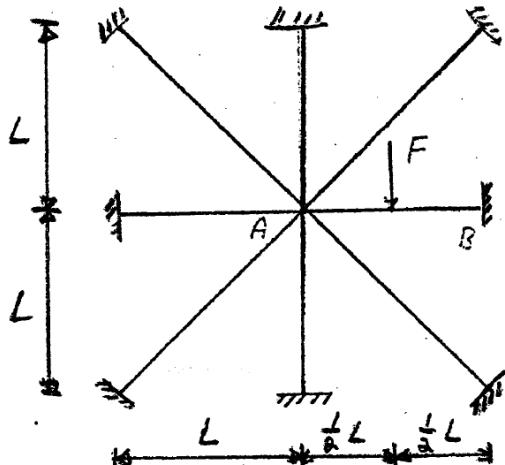
Note for finnish students.

Suomessa on joskus tapana piirtää taivutusmomenttikuvio alla olevan kuvan mukaisesti. Huomaa taivutusmomenttiakselin positiivinen suunta. Palataan rasituskuvioiden merkkisääntöihin myöhemmin.



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1. Calculate the rotation of the middle joint for the frame shown. The flexural rigidity of the beams is the same EI . Draw the bending-moment diagram for the beam AB.

Poistamalla translaatiointiirymät jokaiselta elementiltä huomataan, että kunkin elementin jäykkyysmatriisi on muotoa

$$\mathbf{k}_i = \frac{EI}{L_i} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\mathbf{k}_e = \frac{EI_z}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Ainut globaaliiriitmä Q on pisteen A kiertymä, jonka jäykkyys

$$\mathbf{K} = 4 \cdot 4 \frac{EI}{L} + 4 \cdot 4 \frac{EI}{\sqrt{2}L} = 8 \cdot (2 + \sqrt{2}) \frac{EI}{L}$$

Elementin AB ekvivalenttinen solmukuormitusvektori on $\mathbf{f}_{AB}^p = \frac{FL}{8} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, joten globaali kuormitusvektori

$$\mathbf{F} = -\frac{FL}{8}$$

$$\text{Pisteen A kiertymä } \mathbf{Q}_A = -\frac{1}{8 \cdot (2 + \sqrt{2})} \frac{L}{EI} \frac{FL}{8} = -\frac{FL^2}{64 \cdot (2 + \sqrt{2}) EI} \approx -\frac{FL^2}{218.51 \cdot EI}$$

Lasketaan sitten elementin AB solmuvoimat käyttäen neljän vapausasteen elementtiä, jotta saadaan mukaan myös pystysuuntaiset voimat

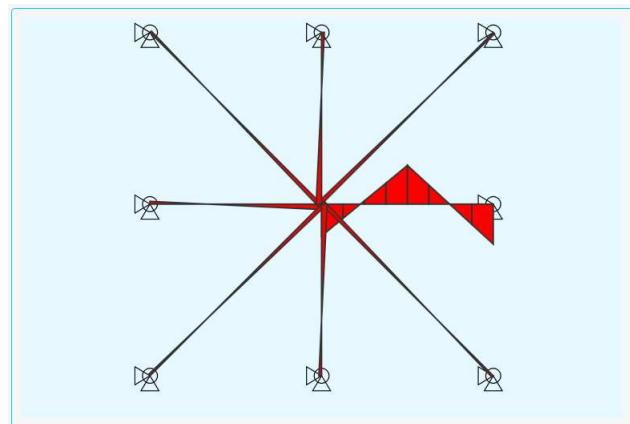
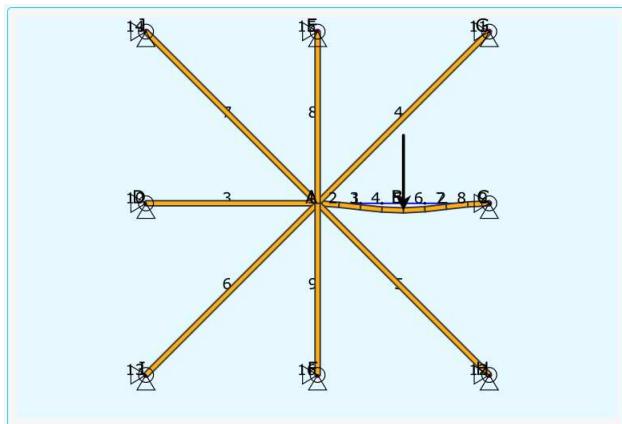
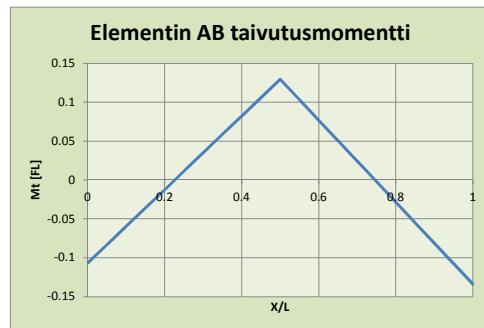
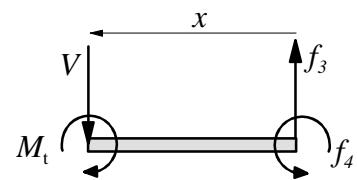
$$\mathbf{f}_1 = \mathbf{k}_1 \mathbf{q}_1 - \mathbf{f}_1^p = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \left(\frac{FL^2}{218.51 \cdot EI} \right) \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} F/2 \\ FL/8 \\ F/2 \\ -FL/8 \end{bmatrix} = F \begin{bmatrix} 0.473 \\ 0.107L \\ 0.527 \\ -0.134L \end{bmatrix}$$

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kuvassa on elementin A-B oikeanpuoleinen osuuus ennen pistekuormitusta F . Momenttitasapainosta pisteen x suhteeseen saadaan

$$\text{at } x \quad +f_3 x + f_4 - M_t(x) = 0 \quad \Rightarrow \\ M_t(x) = +f_3 x + f_4 \quad \Rightarrow \\ M_t(L/2) = +0.527 F \frac{L}{2} - 0.134 \cdot FL = +0.1295 \cdot FL$$

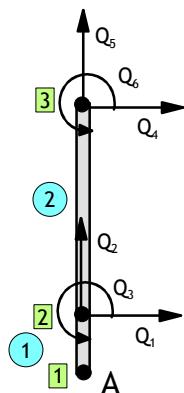


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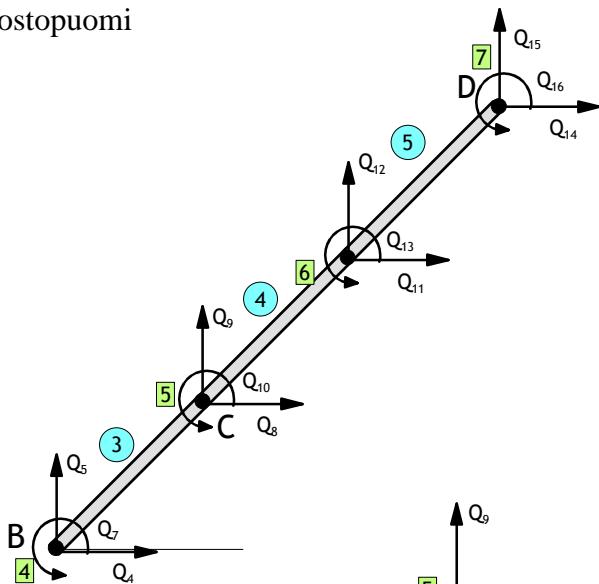
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Tehtävä 3

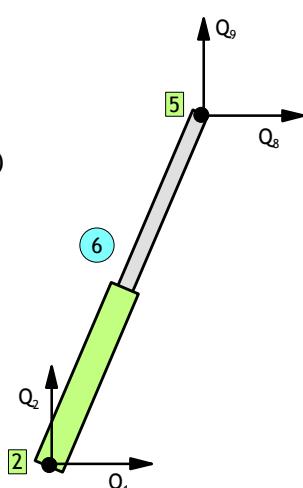
Pylväs



Nostopuomi



Nostosylinteri (sauva)



Elementtien solmut		
Elem	Node1	Node2
1	1	2
2	2	3
3	4	5
4	5	6
5	6	7
6	2	5

ID-taulukko			
Solmu	Qx	Qy	Rz
1	0	0	0
2	1	2	3
3	4	5	6
4	4	5	7
5	8	9	10
6	11	12	13
7	14	15	16

Theta	40	ast	0.698132	rad
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Nostosylinteri

x21	766.0444	mm	=1000*COS(Theta)
y21	1642.788	mm	=1000+1000*SIN(Theta)
Ls	1812.616	mm	
I	0.422618		Suuntakosinit
m	0.906308		
E	200000	MPa	
A	4000	mm ²	
EA/L	441351.2	N/mm	

1	2	8	9	ID
78828	169047	-78828	-169047	1
169047	362523	-169047	-362523	2
-78828	-169047	78828	169047	8
-169047	-362523	169047	362523	9

Mallin vapausasteiden lukumäärä on 16 (ID-taulukon max.). Elementin 6 solmu 1 on solmu nr. 2, joten jäykkyysmatriisiin kaksi ens. riviä ja saraketta menevät globaalista jäykkyysmatriisiin riveille ID(2,1)=1 ja ID(2,2)=2. Elementin 6 solmu 2 on solmu nr. 5, joten jäykkyysmatriisiin rivit 3 ja 4 ja sarakkeet 3 ja 4 menevät globaalista jäykkyysmatriisiin riveille ID(5,1)=8 ja ID(5,2)=9 ja sarakkeille.

$$K(ID(2,1), ID(2,1)) = K(ID(2,1), ID(2,1)) + k(1,1) \text{ eli } K(1,1)$$

$$K(ID(2,1), ID(2,2)) = K(ID(2,1), ID(2,2)) + k(1,2) \text{ eli } K(1,2)$$

$$K(ID(2,1), ID(5,1)) = K(ID(2,1), ID(5,1)) + k(1,3) \text{ eli } K(1,8)$$

$$K(ID(2,1), ID(5,2)) = K(ID(2,1), ID(5,2)) + k(1,4) \text{ eli } K(1,9)$$

jne.

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1	2	8	9 ID
78828	169047	-78828	-169047 1
169047	362523	-169047	-362523 2
-78828	-169047	78828	169047 8
-169047	-362523	169047	362523 9

Katsotaan vielä tilannetta, jossa globaalii jäykkyysmatriisiin \mathbf{K} on sijoittelusummattu ainoastaan sauvan jäykkyysmatriisi