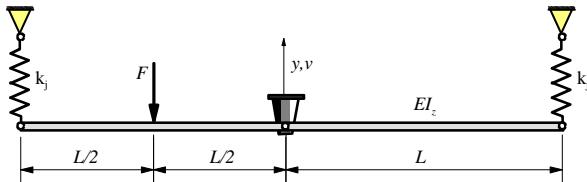


## Exam II 21.1.2014 Solutions

NOTE. Due to a trip to abroad the inspection of this exam starts 10 Feb 2014. Results should be available few days after that.



1. Compute the deflection at right spring using the potential energy  $\Pi = U + WP$  minimum principle. Choose kinematically admissible trial function from the complete set  $p(x) = \alpha_0 + \alpha_1\left(\frac{x}{L}\right) + \alpha_2\left(\frac{x}{L}\right)^2$ . Given values:  $E = 200$  GPa,  $I_z = 5 \cdot 10^6$  mm $^4$ ,  $k_j = 1000$  N/mm,  $F = 10000$  N and  $L = 2000$  mm.

Boundary condition:

$$\tilde{v}(0) = 0 \Rightarrow \tilde{v}(x) = \alpha_1\left(\frac{x}{L}\right) + \alpha_2\left(\frac{x}{L}\right)^2 \Rightarrow \tilde{v}(L) = \alpha_1 + \alpha_2$$

$$\tilde{v}_{,x} = \frac{\alpha_1}{L} + 2\frac{\alpha_2}{L^2}x; \quad \tilde{v}_{,xx} = 2\frac{\alpha_2}{L^2}; \quad \tilde{v}_{,xxx} = 4\frac{\alpha_2}{L^4}$$

Beam strain energy

$$U_p = \frac{EI_z}{2} \int_{-L}^L \tilde{v}_{,xx}(x)^2 dx = \frac{EI_z}{2} \cdot 4 \frac{\alpha_2^2}{L^4} \int_{-L}^L dx = \frac{4EI_z}{L^3} \alpha_2^2$$

Strain energy of springs

$$U_j = \frac{1}{2} k_j \tilde{v}(-L)^2 + \frac{1}{2} k_j \tilde{v}(L)^2 = \frac{1}{2} k_j [(\alpha_2 - \alpha_1)^2 + (\alpha_2 + \alpha_1)^2] = k_j [\alpha_1^2 + \alpha_2^2]$$

Work Potential (force towards negative direction)

$$WP = F\tilde{v}\left(\frac{-L}{2}\right) = \frac{F}{4}(\alpha_2 - 2\alpha_1)$$

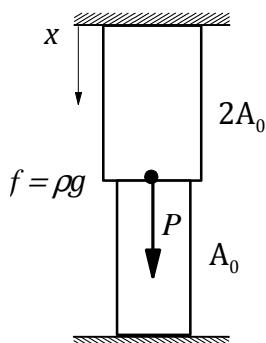
Potential energy minimum principle  $\Pi = U_j + U_p + WP \rightarrow \min \Rightarrow$

$$\begin{cases} \frac{\partial U}{\partial \alpha_1} = -\frac{\partial WP}{\partial \alpha_1} \\ \frac{\partial U}{\partial \alpha_2} = -\frac{\partial WP}{\partial \alpha_2} \end{cases} \Rightarrow \begin{cases} 2k_j \alpha_1 = \frac{F}{2} \\ 2k_j \alpha_2 + 8\frac{EI_z}{L^3} \alpha_2 = -\frac{F}{4} \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{F}{4k_j} \\ \alpha_2 = \frac{-FL^3}{8k_j L^3 + 32EI_z} \end{cases}$$

Given value substitution gives

F	10000	N
$k_j$	1000	N/mm
E	200000	MPa
$I_z$	5.00E+06	mm $^4$
L	2000	mm

$\alpha_1$	2.5	mm
$\alpha_2$	-0.83333	mm
v(L)	1.666667	mm



2. Define the nodal displacement and sketch the normal force diagram of the rod structure depicted on the left. Compute also the support reactions on the top and bottom. In addition to the gravity (downwards), a point load  $P = 10000$  N is acting to the structure. The length of the both rod members is  $L = 10$  m, density =  $8000$  kg/m<sup>3</sup>. The cross-sectional area  $A_0$  is  $10000$  mm<sup>2</sup>. Young's modulus is  $200$  GPa. Use  $10$  m/s<sup>2</sup> for the standard earth gravity.

Element stiffness matrices and equivalent nodal load vectors

$$\mathbf{k}_1 = \frac{E \cdot 2A_0}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{f}_1^V = \frac{\rho g \cdot 2A_0 L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8000 \\ 8000 \end{bmatrix} N \quad G_1 = 16000 N$$

$$\mathbf{k}_2 = \frac{EA_0}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{f}_2^V = \frac{\rho g A_0 L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4000 \\ 4000 \end{bmatrix} N \quad G_2 = 8000 N$$

Global stiffness, load and displacement

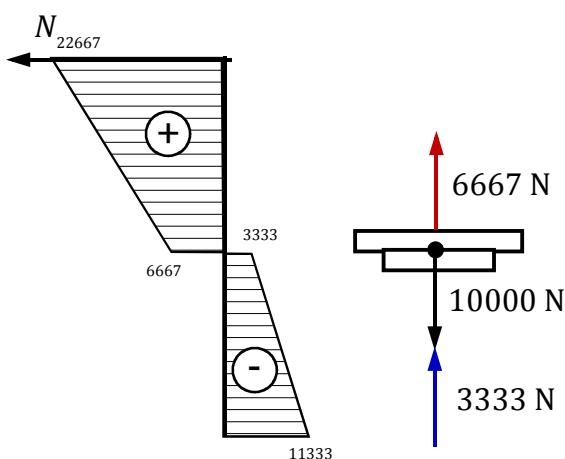
$$\mathbf{K} = \frac{3EA_0}{L} = 6 \cdot 10^8 \frac{N}{m} \quad \mathbf{F} = P + \mathbf{f}_{12}^V + \mathbf{f}_{21}^V = 22000 N \quad \mathbf{Q} = \mathbf{K}^{-1} \mathbf{F} = 3.667 \cdot 10^{-5} m$$

Element nodal forces

$$\mathbf{f}_1 = \mathbf{k}_1 \cdot \begin{bmatrix} 0 \\ Q \end{bmatrix} - \mathbf{f}_1^V = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} 10^8 \cdot \begin{bmatrix} 0 \\ 3.667 \end{bmatrix} 10^{-5} - \begin{bmatrix} 8000 \\ 8000 \end{bmatrix} = \begin{bmatrix} -22667 \\ 6667 \end{bmatrix} N$$

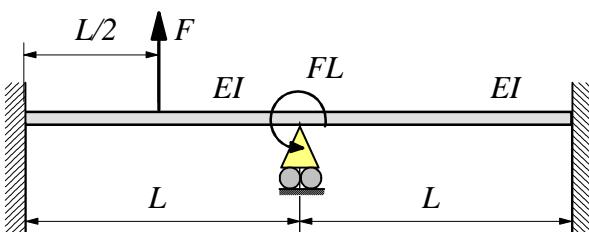
$$\mathbf{f}_2 = \mathbf{k}_2 \cdot \begin{bmatrix} Q \\ 0 \end{bmatrix} - \mathbf{f}_2^V = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} 10^8 \cdot \begin{bmatrix} 3.667 \\ 0 \end{bmatrix} 10^{-5} - \begin{bmatrix} 4000 \\ 4000 \end{bmatrix} = \begin{bmatrix} 3333 \\ -11333 \end{bmatrix} N \quad N_1 = -f_1 \quad N_2 = f_2$$

Nodal normal forces: Element 1 (top)  $N_1 = +22667$ ,  $N_2 = +6667$ , Element2 (bottom)  $N_1 = -3333$ ,  $N_2 = -11333$



Support reactions (top)  $T_y = -22667$  N (bottom)  
 $T_a = -11333$  N. Equilibrium:  $+T_y + T_a + G_1 + G_2 + P = 0$  N  
OK

## Exam II 21.1.2014 Solutions



3. Draw the bending moment diagram of the beam structure on the left. Use two beam elements with two nodal degrees of freedom.

Element 1 (left beam)

$$\mathbf{k}_1 = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{f}_1^p = \begin{bmatrix} F/2 \\ FL/8 \\ F/2 \\ -FL/8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Element 2 (right beam)

$$\mathbf{k}_2 = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Global stiffness matrix  $\mathbf{K}$ , global load vector  $\mathbf{F}$  and global displacement vector  $\mathbf{Q}$

$$\mathbf{K} = \frac{8EI}{L} \Rightarrow \mathbf{K}^{-1} = \frac{L}{8EI} \quad \mathbf{F} = FL - FL/8 = 7FL/8$$

$$\mathbf{Q} = \mathbf{K}^{-1}\mathbf{F} = \frac{L}{8EI} \cdot \frac{7FL}{8} = \frac{7FL^2}{64EI}$$

Element 1 nodal force vector

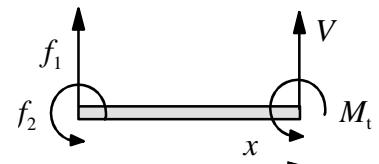
$$\mathbf{f}_1 = \mathbf{k}_1 \mathbf{q}_1 = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{FL^2}{64EI} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \end{bmatrix} - \frac{F}{8} \begin{bmatrix} 4 \\ L \\ 4 \\ -L \end{bmatrix} = \frac{F}{32} \begin{bmatrix} 5 \\ 3L \\ -37 \\ 18L \end{bmatrix}$$

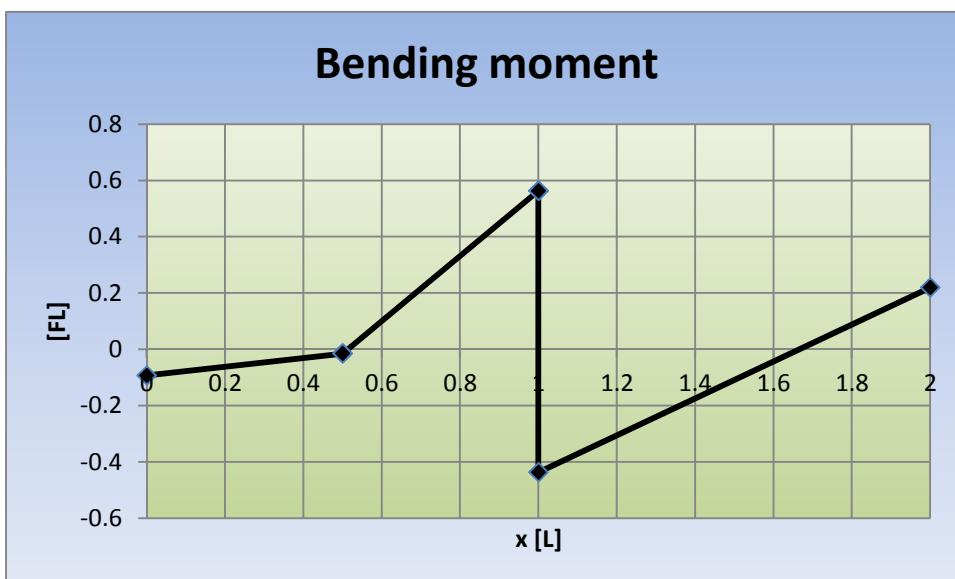
Element 2 nodal force vector

$$\mathbf{f}_2 = \mathbf{k}_2 \mathbf{q}_2 = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{FL^2}{64EI} \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{F}{32} \begin{bmatrix} 21 \\ 14L \\ -21 \\ 7L \end{bmatrix}$$

Free body diagram of the element 1 first part (before the point load)

$$-f_1 x + f_2 + M_t = 0 \Rightarrow M_t = f_1 x - f_2 \Rightarrow M_t(L/2) = \frac{5FL}{64} - \frac{6FL}{64} = -\frac{FL}{64}$$

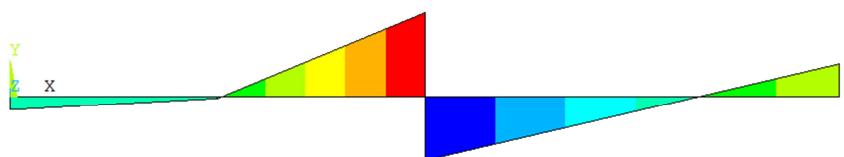


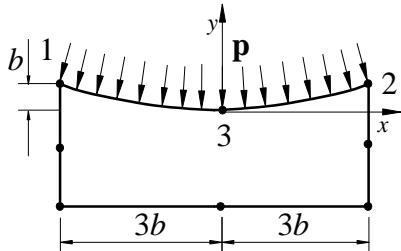


## ANSYS check

```
LINE STRESS
STEP=1
SUB =1
TIME=1
SMIS6   SMIS12
MIN =-.4375
ELEM=3
MAX =.5625
ELEM=2
```

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19:20:36





4. Calculate numerically the length of the line 1-3-2 of a quadratic element left. Use three integration points. Length  $b$  is 100 mm.

$$N_1^s = -\frac{1}{2}\xi(1-\xi)$$

$$J_x = N_{1,\xi}x_1 + N_{2,\xi}x_2 N_{3,\xi}x_3$$

$$J_y = N_{1,\xi}y_1 + N_{2,\xi}y_2 N_{3,\xi}y_3$$

$$N_2^s = \frac{1}{2}\xi(1+\xi)$$

$$N_3^s = 1 - \xi^2$$

$$x_s = N_1^s x_2 + N_2^s x_2 + N_3^s x_3 = \frac{3b}{2} [\xi - \xi^2 + \xi + \xi^2] = 3b\xi \quad \Rightarrow dx = 3b d\xi = J_x d\xi$$

$$y_s = N_1^s y_1 + N_2^s y_2 + N_3^s y_3 = \frac{b}{2} [-\xi + \xi^2 + \xi + \xi^2] = b\xi^2 \quad \Rightarrow dy = 2b\xi d\xi = J_y d\xi$$

$$\Rightarrow ds = \sqrt{J_x^2 + J_y^2} d\xi = b\sqrt{4\xi^2 + 9} d\xi$$

$\xi$	Jx	Jy	Sqr(Jx <sup>2</sup> +Jy <sup>2</sup> )	Wi	li
-0.7746	300	-154.919	337.63886	0.555556	187.5771
0.0000	300	0	300	0.888889	266.6667
0.7746	300	154.9193	337.63886	0.555556	187.5771
<b>TOTAL</b>					<b>641.821</b>

Check

$$b := 100 \quad x1 := -3 \cdot b \quad x2 := 3 \cdot b \quad x3 := 0 \\ y1 := b \quad y2 := b \quad y3 := 0$$

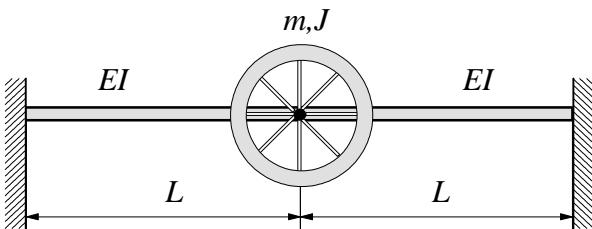
$$N1(\xi) := \frac{-\xi}{2} \cdot (1 - \xi) \quad N2(\xi) := \frac{\xi}{2} \cdot (1 + \xi) \quad N3(\xi) := 1 - \xi^2$$

$$x(\xi) := N1(\xi) \cdot x1 + N2(\xi) \cdot x2 + N3(\xi) \cdot x3 \quad y(\xi) := N1(\xi) \cdot y1 + N2(\xi) \cdot y2 + N3(\xi) \cdot y3$$

$$Jx := \frac{d}{d\xi} x(\xi) \rightarrow 300 \quad Jy(\xi) := \frac{d}{d\xi} y(\xi) \rightarrow 200 \cdot \xi$$

$$\text{Length} := \int_{-1}^1 \sqrt{Jx^2 + Jy(\xi)^2} d\xi \rightarrow 450 \cdot \ln(\sqrt{13} + 2) - 450 \cdot \ln(3) + 100 \cdot \sqrt{13}$$

$$\text{Length} = 641.87$$



5. Define the two lowest natural frequencies of the combined beam/mass system on the left. Sketch also the comparable eigen vectors. Use two beam elements with two nodal degrees of freedom and consistent mass matrix. Mass  $m$  is 1000 kg and moment of inertia  $J = 1000 \text{ kgm}^2$ .

Var.	Value	Unit
$\rho$	7850	$\text{kg}/\text{m}^3$
$E$	200	GPa
$I_z$	$10^{-5}$	$\text{m}^4$
$A$	0.01	$\text{m}^2$
$L$	2	m

$$\mathbf{k}_e = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\mathbf{m}_e = \frac{\rho Al_e}{420} \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

Global stiffness matrix  $\mathbf{K}$  (element 1 rows and cols 3,4, element 2 rows and cols 1,2)

$$\mathbf{K} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} = \begin{bmatrix} 6000000 & 0 \\ 0 & 8000000 \end{bmatrix}$$

Global mass matrix  $\mathbf{M}$  (element 1 rows and cols 3,4, element 2 rows and cols 1,2 + flywheel)

$$\mathbf{M} = \frac{\rho AL}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8L^2 \end{bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} = \begin{bmatrix} 1116.6286 & 0 \\ 0 & 1011.9619 \end{bmatrix}$$

$$\det(\mathbf{K} - \lambda \mathbf{M}) = 0 \Rightarrow \begin{vmatrix} 6000 - 1.1166286\lambda & 0 \\ 0 & 8000 - 1.0119619\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\lambda^2 + 1.328 \cdot 10^4 \lambda + 4.248 \cdot 10^7 = 0 \Rightarrow \begin{cases} \lambda_1 = 5373 \\ \lambda_2 = 7905 \end{cases} \Rightarrow \begin{cases} \omega_1 = 73.3 \\ \omega_2 = 88.9 \end{cases} \Rightarrow \begin{cases} f_1 = 11.67 \text{ Hz} \\ f_2 = 14.15 \text{ Hz} \end{cases}$$

Eigen vectors

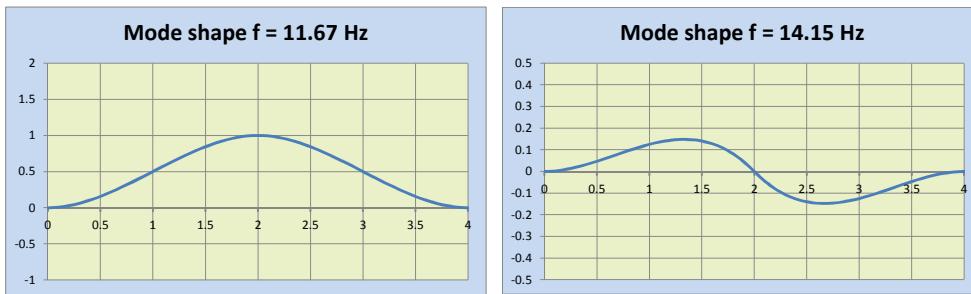
$$(\mathbf{K} - \lambda_1 \mathbf{M}) \mathbf{x}^1 = \mathbf{0} \Rightarrow \begin{bmatrix} 6000000 - 5373.318 \cdot 1116.629 & 0 \\ 0 & 8000000 - 5373.318 \cdot 1011.962 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 2562407 \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2^1 = 0 \quad x_1^1 = 1$$

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$$(\mathbf{K} - \lambda_2 \mathbf{M}) \mathbf{x}^2 = \mathbf{0} \Rightarrow \begin{bmatrix} 6000000 - 7905.436 \cdot 1116.629 & 0 \\ 0 & 8000000 - 7905.436 \cdot 1011.962 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} -2827436 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1^2 = 0 \quad x_2^2 = 1$$



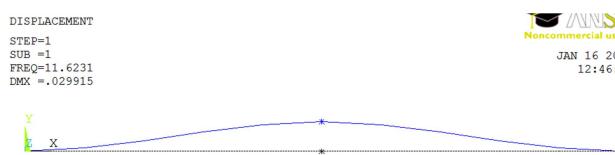
Check (MathCad)

$$\frac{\sqrt{\text{genvals}(\text{Kalle}, \text{Massa})}}{2 \cdot \pi} = \begin{pmatrix} 11.667 \\ 14.151 \end{pmatrix}$$

$$\text{genvecs}(\text{Kalle}, \text{Massa}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Check (ANSYS Classic)

```
SET TIME/FREQ LOAD STEP SUBSTEP CUMULATIVE
1 11.623      1   1   1
2 14.112      1   2   2
```



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DISPLACEMENT  
STEP=1  
SUB =2  
FREQ=14.1123  
DMX = .009128

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12:47:52

Note. There are 10 Beam 188 elements in Ansys model in order to plot modes with reasonable accuracy.