Introduction to materials modelling

6. exercise – transversely isotropic elastic material model

1. Let $\boldsymbol{A}$ be a symmetric second order tensor and $\boldsymbol{m}$ a vector. Show that the invariant $I_4 = \boldsymbol{m}\cdot\boldsymbol{A}\boldsymbol{m} = m_i A_{ij} m_j$ can be written in the form $I_4 = \mathrm{tr}(\boldsymbol{A}\boldsymbol{M})$, where $\boldsymbol{M} = \boldsymbol{m}\otimes\boldsymbol{m}$, in component form $M_{ij} = m_i m_j$.

2. As in the previous problem $\boldsymbol{A}$ is a symmetric second order tensor and $\boldsymbol{M} = \boldsymbol{m}\otimes\boldsymbol{m}$ is a structural tensor where $\boldsymbol{m}$ is a unit vector. The invariant basis containing $\boldsymbol{m}$ and $\boldsymbol{A}$ consist of elements

$$I_1 = \mathrm{tr}\boldsymbol{A}, \quad I_2 = \frac{1}{2}\mathrm{tr}(\boldsymbol{A}^2), \quad I_2 = \frac{1}{3}\mathrm{tr}(\boldsymbol{A}^3), \quad I_4 = \mathrm{tr}(\boldsymbol{A}\boldsymbol{M}), \quad I_5 = \mathrm{tr}(\boldsymbol{A}^2\boldsymbol{M}). \tag{1}$$

   Show that the invariant $\tilde{I} = \mathrm{tr}(\boldsymbol{A}^3\boldsymbol{M})$ can be expressed in terms of the base (1).

   Hint. Make use of the Cayley-Hamilton theorem which says that the tensor itself satisfies its characteristic polynomial

$$-\lambda^3 + \hat{I}_1\lambda^2 + \hat{I}_2\lambda + \hat{I}_3 = 0,$$

   where $\hat{I}_i$ are the principal invariants of $\boldsymbol{A}$:

$$\hat{I}_1 = \mathrm{tr}\boldsymbol{A}, \quad \hat{I}_2 = \frac{1}{2}\left[\mathrm{tr}(\boldsymbol{A}^2) - (\mathrm{tr}\boldsymbol{A})^2\right], \quad \hat{I}_3 = \det\boldsymbol{A}.$$

3. Consider a unidirectionally fibre reinforced material. Assume for simplicity that the cross-section of a fibre is square, having side length of $2b$. The fibres are arranged in a regular grid where the distance between the centroids of the fibres is $2a$. The cross-section area of a single fibre is $A_\mathrm{f} = 4b^2$ and the representative are is $A_\mathrm{r} = 4a^2$. Material of both the fibres and the matrix is assumed to be isotropic linear elastic having Young's modulus and Poisson's ratio E_f and ν_f for the fibre, respectively and $E_\mathrm{m}, \nu_\mathrm{m}$ for the matrix. The volume ratio of the fibres is $f = (b/a)^2$ (same as the area ratio). Determine, if possible, the elastic coefficients E_L, E_T, G_L, ν_L and ν_T for the homogenized transversely isotropic solid expressed in terms of the elastic coefficients of the fibres and matrix and the volume ratio.

   Hint. Examine a cubic representative volume element (RVE). Set either constant strain or stress state acting to the RVE in proper directions and calculate the average quantities. For example, set a longitudinal strain and calculate the force in the cross-section. The average stress is obtained by dividing the force with the cross-section area.

4. For a graphite-epoxy laminate the following material parameters have been measured: in the transverse isotropy plane $E_T = 9,65$ GPa, and the Poisson's ratio $\nu_T = 0,6$ and in the longitudinal direction: $E_L = 148$ GPa, $G_L = 4,55$ GPa, and $\nu_L = 0,3$. Investigate if the given material parameter set is thermodynamically admissible. Does it fulfil the monotonicity condition?