FEM advanced course

Lecture 2 - Kinematics, balance equations, stress measures, linearization

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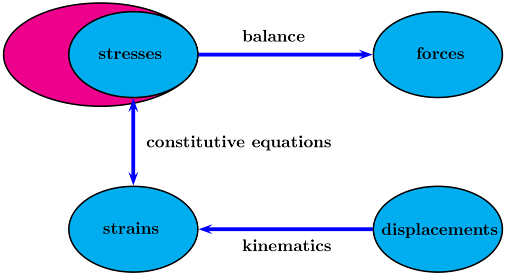
General structure



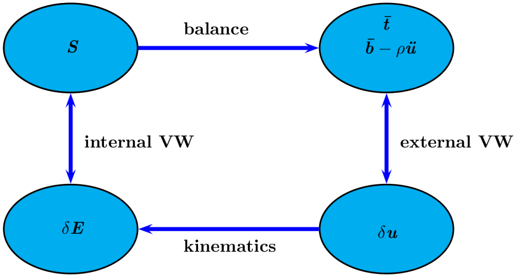
$$B^*\boldsymbol{\sigma} = \rho\bar{\boldsymbol{b}}$$ equilibrium

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}$$ constitutive model

$$\boldsymbol{\varepsilon} = G\boldsymbol{u}$$ kinematical relation in linear case $G = B$

B^* is the adjoint operator of B.

Principle of virtual work



$$-\int_{\Omega_0} \delta\boldsymbol{E} : \boldsymbol{S}\,\mathrm{d}V + \int_{\Omega_0} \delta\boldsymbol{u}\cdot\rho_0\bar{\boldsymbol{b}}\,\mathrm{d}V + \int_{\partial\Omega_{t0}} \delta\boldsymbol{u}\cdot\bar{\boldsymbol{t}}\,\mathrm{d}A - \int_{\Omega_0} \delta\boldsymbol{u}\cdot\ddot{\boldsymbol{u}}\rho_0\,\mathrm{d}V = 0$$

$B^*\boldsymbol{S} = \rho_0\bar{\boldsymbol{b}}$ equilibrium

$\boldsymbol{S} = \mathbb{C}\boldsymbol{E}$ constitutive model

$\boldsymbol{E} = G\boldsymbol{u} \quad \Rightarrow \quad \delta\boldsymbol{E} = B\delta\boldsymbol{u}$ kinematical relation

B^* is the adjoint operator of B.

Description of motion

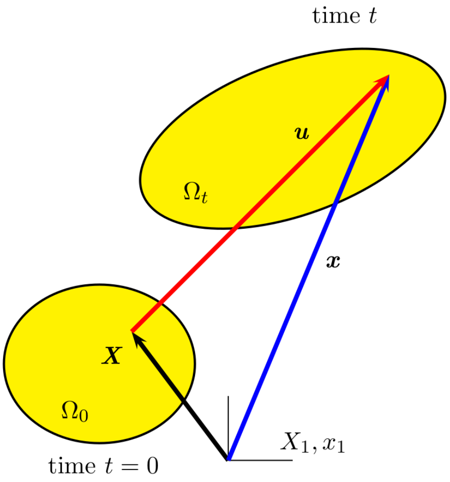
A material point has coordinates $\boldsymbol{X}$ in the undeformed state.
After deformation it is moved to the place $\boldsymbol{x}$. A mapping $\boldsymbol{\varphi}$ is called the motion

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)$$
$$x_i = \varphi_i(\boldsymbol{X}, t) = X_i + u_i(\boldsymbol{X}, t),$$

and $\boldsymbol{u}$ is the displacement vector.

- $\boldsymbol{X}$ are the material coordinates. It means that $\boldsymbol{X}$ indicates the position of a **material point at the initial configuration**. Frequently used in solid mechanics.
- $\boldsymbol{x}$ are the spatial coordinates. Much used in fluid mechanics.

This distinction is important.



Deformation gradient

Deformation gradient $\boldsymbol{F}$ gives the change of an infinitesimal line element at P

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\mathrm{d}\boldsymbol{X}, \qquad \boldsymbol{F} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}},$$

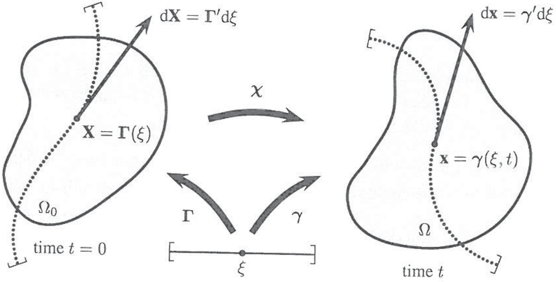


Figure from G.Holzapfel: *Nonlinear solid mechanics*, p. 70

Deformation gradient - cont'd

In indicial notation

$$\mathrm{d}x_i = F_{ij}\mathrm{d}X_j,$$

$$F_{ij} = \frac{\partial \varphi_i}{\partial X_j} = \frac{\partial X_i}{\partial X_j} + \frac{\partial u_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}, \quad \text{or} \quad \boldsymbol{F} = \boldsymbol{I} + \boldsymbol{H}, \quad \text{where} \quad \boldsymbol{H} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}$$

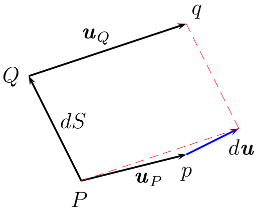
is the displacement gradient. If there is no deformation, then $\boldsymbol{F} = \boldsymbol{I}$.
Deformation gradient $\boldsymbol{F}$ contains both strains and rigid body rotation and can be decomposed as (the polar decomposition)

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R},$$

where $\boldsymbol{R}$ is orthogonal rotation tensor and $\boldsymbol{U}$ and $\boldsymbol{V}$ are the symmetric and positive definite *right and left stretch tensors*.

Definition of strain

Length of a line element PQ is $\mathrm{d}S = \sqrt{\mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}}$
In deformed state $|pq| = \mathrm{d}s = \sqrt{\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}}$



$$
\begin{aligned}
\tfrac{1}{2}[(\mathrm{d}s)^2 - (\mathrm{d}S)^2] &= \tfrac{1}{2}(\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}) \\
&= \tfrac{1}{2}\mathrm{d}\boldsymbol{X}\cdot(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})\mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{X}\cdot\boldsymbol{E}\,\mathrm{d}\boldsymbol{X}
\end{aligned}
$$

where $\boldsymbol{E}$ is the Green-Lagrange strain tensor.

Green-Lagrange strain tensor

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}),$$

where $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^2$ is the *right Cauchy-Green deformation tensor* and $\boldsymbol{U}$ is the *right Cauchy-Green stretch tensor*. For pure rigid body rotation $\boldsymbol{E} = \boldsymbol{0}$.
G-L in terms of displacement

$$\boldsymbol{E} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right) = \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T + \boldsymbol{H}^T\boldsymbol{H}\right)$$

If $\partial \boldsymbol{u}/\partial \boldsymbol{X} \ll 1$, then

$$\boldsymbol{E} \approx \boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T\right) = \frac{1}{2}\left(\boldsymbol{H} + \boldsymbol{H}^T\right) = \mathrm{sym}\,\mathrm{grad}\,\boldsymbol{u},$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor - notice that in geometrically linear theory $\boldsymbol{x} = \boldsymbol{X}$.

Other strain tensors

A general strain definition can be stated as

$$\boldsymbol{E}^{(m)} = \frac{1}{m}\left(\boldsymbol{U}^m - \boldsymbol{I}\right) \quad \text{or} \quad \boldsymbol{e}^{(m)} = \frac{1}{m}\left(\boldsymbol{V}^m - \boldsymbol{I}\right).$$

- $m = 2$ corresponds to the G-L strain tensor.
- The Hencky or logarithic strain tensor is obtained when $m \to 0^+$

$$\lim_{m\to 0^+} \boldsymbol{E}^{(m)} = \ln \boldsymbol{U}, \quad \text{or} \quad \lim_{m\to 0^+} \boldsymbol{e}^{(m)} = \ln \boldsymbol{V},$$

- The Biot strain tensor for $m = 1$

$$\boldsymbol{E}^{(1)} = \boldsymbol{U} - \boldsymbol{I}.$$

Some people call the logarithmic strain as the true strain. Please, do not use that naming. Definition of a strain is a geometrical concept and all properly defined strain measures describe strain state correctly.

For interested reader, more on strain measures can be found in Finnish at http://rmseura.tkk.fi/rmlehti/2016/nro2/RakMek_49_2_2016_6.pdf

Other strain tensors (cont'd)

Spatial (Eulerian) strain tensor when $m = -2$ is called the Almansi strain tensor (or Almansi-Hamel)

$$\boldsymbol{e}^{(-2)} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{V}^{-2}\right) = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{b}^{-1}\right),$$

where $\boldsymbol{b}$ is the *left Cauchy-Green deformation tensor tensor* and it is related to the *left Cauchy-Green stretch tensor* $\boldsymbol{V}$ as

$$\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{V}\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{V}^T = \boldsymbol{V}^2.$$

Tensors $\boldsymbol{C}, \boldsymbol{U}, \boldsymbol{b}$ and $\boldsymbol{V}$ are frequently used in large strain elastic constitutive models.

Infinitesimal strain tensor

If deformations (displacements, rotations) are small, distinction between material and spatial coordinates is irrelevant.

Infinitesimal strain tensor, also known as the small strain tensor is defined as

$$\boldsymbol{\varepsilon} = \operatorname{sym}\operatorname{grad}\boldsymbol{u}$$

or in index notation

$$\varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Von Kármán notation

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}$$

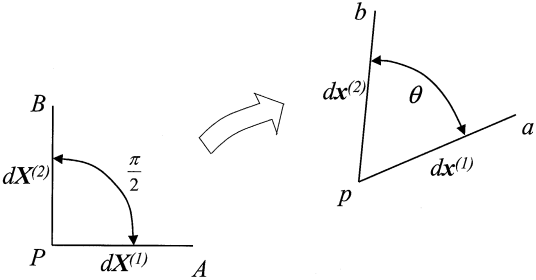
Strain in arbitrary direction

Strain in direction $\boldsymbol{n}$ ($|\boldsymbol{n}| = 1$)

$$\varepsilon_n = \boldsymbol{n}\cdot\boldsymbol{\varepsilon}\boldsymbol{n}.$$

Change in the angle between orthonormal vectors $\boldsymbol{n}$ and $\boldsymbol{m}$

$$\gamma_{nm} = 2\boldsymbol{n}\cdot\boldsymbol{\varepsilon}\boldsymbol{m}.$$



Principal strains

Eigenvalues of the strain tensor

$$\boldsymbol{\varepsilon}\boldsymbol{n} = \lambda\boldsymbol{n} \qquad (\boldsymbol{\varepsilon} - \lambda\boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}$$

Non-trivial solution for $\boldsymbol{n}$ if

$$\det(\boldsymbol{\varepsilon} - \lambda\boldsymbol{I}) = 0$$

Characteristic polynomial

$$-\lambda^3 + I_1^\varepsilon\lambda^2 + I_2^\varepsilon\lambda + I_3^\varepsilon = 0$$

where

$$\begin{aligned} I_1^\varepsilon &= \mathrm{tr}\boldsymbol{\varepsilon} = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\ I_2^\varepsilon &= \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{\varepsilon}^2) - (\mathrm{tr}\boldsymbol{\varepsilon})^2] \\ I_3^\varepsilon &= \det\boldsymbol{\varepsilon} \end{aligned}$$

are called the *principal invariants* of the infinitesimal strain tensor.

Principal stretches

Eigenvalues λ of the right stretch tensor

$$\boldsymbol{U}\boldsymbol{n} = \lambda\boldsymbol{n} \qquad (\boldsymbol{U} - \lambda\boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}$$

Non-trivial solution for $\boldsymbol{n}$ if

$$\det(\boldsymbol{U} - \lambda\boldsymbol{I}) = 0$$

Characteristic polynomial

$$-\lambda^3 + I_1^U\lambda^2 + I_2^U\lambda + I_3^U = 0$$

where

$$I_1^U = \mathrm{tr}\,\boldsymbol{U}, \quad I_2^U = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{U}^2) - (\mathrm{tr}\,\boldsymbol{U})^2] \quad I_3^U = \det\boldsymbol{U}$$

are called the *principal invariants* of the strech tensor and $\lambda_1, \lambda_2, \lambda_3$ are the *principal stretches*.

Volumetric - isochoric split in small strains

The small strain tensor can be *additively* split into volumetric and isochoric i.e. volume preserving parts as

$$\boldsymbol{\varepsilon} = \tfrac{1}{3}(\mathrm{tr}\boldsymbol{\varepsilon})\boldsymbol{I} + \boldsymbol{e}$$

where $\mathrm{tr}\boldsymbol{\varepsilon} = \varepsilon_{\mathrm{vol}}$ is the volumetric strain

$$\varepsilon_{\mathrm{vol}} = \frac{V - V_0}{V_0},$$

and $\boldsymbol{e}$ is the deviatoric part of the strain tensor ($\mathrm{tr}\boldsymbol{e} = 0$).

Volumetric - isochoric split for large strains

In large strain analysis, the deformation gradient $\boldsymbol{F}$ is *multiplicatively* decomposed into *volume changing i.e. dilatational* and *volume preserving i.e. distortional* parts. Relative volume change is $J = \det \boldsymbol{F}$, thus

$$\boldsymbol{F} = (J^{1/3}\boldsymbol{I})\hat{\boldsymbol{F}} = J^{1/3}\hat{\boldsymbol{F}}, \quad \text{also} \quad \boldsymbol{C} = (J^{2/3}\boldsymbol{I})\hat{\boldsymbol{C}} = J^{2/3}\hat{\boldsymbol{C}}$$

Now $\det \hat{\boldsymbol{F}} = 1$ and $\det \hat{\boldsymbol{C}} = (\det \hat{\boldsymbol{F}})^2 = 1$.

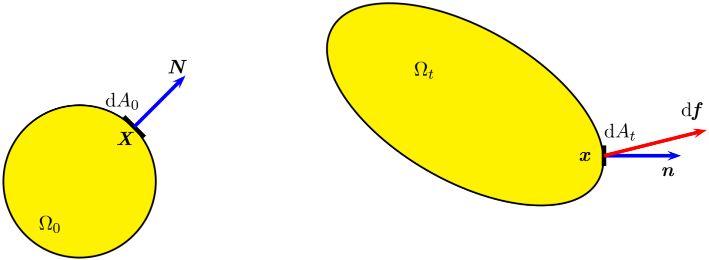
Logarithmic strains decompose additively! We will return to this later.

Cauchy stress tensor

The Cauchy stress tensor $\boldsymbol{\sigma}$ gives the actual force $\mathrm{d}\boldsymbol{f}$ on the deformed surface area $\mathrm{d}A_t$ on the deformed configuration at $\boldsymbol{x}$

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{\sigma}\boldsymbol{n}\mathrm{d}A_t,$$

the traction vector is $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$. The Cauchy stress is also called as the true stress.

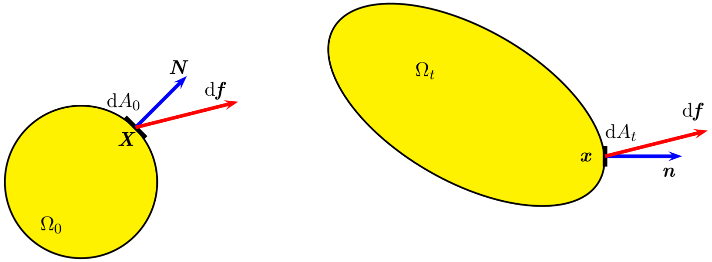


Notice that the indexes of the stress tensor σ_{ij} is now defined such that the first component is in the direction of the stress and the second one to the normal.

Other stress measures, the first Piola-Kirchhoff stress tensor

The first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ gives the actual force $\mathrm{d}\boldsymbol{f}$ on the deformed surface area $\mathrm{d}A_t$, but is reckoned per unit area of the undeformed area $\mathrm{d}A_0$ and expressed the force in terms of the unit normal $\boldsymbol{N}$ to $\mathrm{d}A_0$ at $\boldsymbol{X}$

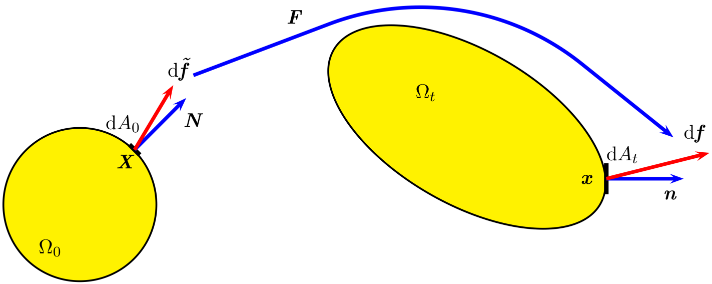
$$\mathrm{d}\boldsymbol{f} = \boldsymbol{\sigma}\boldsymbol{n}\mathrm{d}A_t = \boldsymbol{P}\boldsymbol{N}\mathrm{d}A_0.$$



Other stress measures, the second Piola-Kirchhoff stress tensor

Define a pseudo force vector $\mathrm{d}\tilde{\boldsymbol{f}}$ in the reference configuration such that if we map it with the deformation gradient $\boldsymbol{F}$ we obtain the force vector $\mathrm{d}\boldsymbol{f}$ in the deformed configuration $\mathrm{d}\boldsymbol{f} = \boldsymbol{F}\mathrm{d}\tilde{\boldsymbol{f}}$ or $\mathrm{d}\tilde{\boldsymbol{f}} = \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{f}$, then define the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ as

$$\boldsymbol{S}\boldsymbol{N}\mathrm{d}A_0 = \tilde{\boldsymbol{T}}\mathrm{d}A_0 = \mathrm{d}\tilde{\boldsymbol{f}} = \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{f} = \boldsymbol{F}^{-1}\boldsymbol{P}\boldsymbol{N}\mathrm{d}A_0$$



Relations between different stress tensors

Between Cauchy and PK1

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}, \qquad \boldsymbol{\sigma} = J^{-1}\boldsymbol{P}\boldsymbol{F}^{T}$$

Between PK1 and PK2

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}, \qquad \boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$$

Between Cauchy and PK2

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}, \qquad \boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{T}$$

Cauchy and PK2 stress tensors are symmetric for standard continuum theories (non-polar) but PK1 obeys

$$\boldsymbol{P}\boldsymbol{F}^{T} = \boldsymbol{F}\boldsymbol{P}^{T}$$

Note on dual stress measure

Stress power should be independent of the chosen strain measure. For the Green-Lagrange strain rate $\dot{\boldsymbol{E}}$ the corresponding stress measure is the second Piola-Kirchhoff pseudo-stress $\boldsymbol{S}$ such that the power

$$\int_{\Omega_0} \boldsymbol{S} : \dot{\boldsymbol{E}} \, \mathrm{d}V = \int_{\Omega_t} \boldsymbol{\sigma} : \boldsymbol{D} \, \mathrm{d}v$$

where $\boldsymbol{D}$ is the strain rate tensor, i.e. the symmetric part of the spatial velocity gradient

$$\boldsymbol{D} = \frac{1}{2} \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \right)^T \right)$$

and $\boldsymbol{\sigma}$ is the Cauchy stress tensor (true stress).

In this course we operate in the geometrically linear setting, thus $\dot{\boldsymbol{\varepsilon}} \approx \boldsymbol{D}$, where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor.

Virtual strains and linearization

Virtual G-L strain tensor

$$\delta \boldsymbol{E} = \delta \left[\tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) \right] = \tfrac{1}{2}(\delta \boldsymbol{F}^T \boldsymbol{F} + \boldsymbol{F}^T \delta \boldsymbol{F}).$$

and the virtual deformation gradient is

$$\delta \boldsymbol{F} = \delta(\boldsymbol{I} + \boldsymbol{H}) = \delta \boldsymbol{H} = \frac{\partial \delta \boldsymbol{u}}{\partial \boldsymbol{X}}.$$

Then for the variation of the G-L strain tensor we get

$$\delta \boldsymbol{E} = \tfrac{1}{2}(\delta \boldsymbol{H}^T \boldsymbol{F} + \boldsymbol{F}^T \delta \boldsymbol{H}).$$

For linearized expressions we just change the variation symbol δ to the increment Δ.

Linearization of virtual work

Considering only static case for simplicity

$$-\int_{\Omega_0} \delta \boldsymbol{E} : \boldsymbol{S} \, \mathrm{d}V + \int_{\Omega_0} \delta \boldsymbol{u} \cdot \rho_0 \bar{\boldsymbol{b}} \, \mathrm{d}V + \int_{\partial \Omega_{t0}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A = 0 \tag{1}$$

Assuming constitutive equation in the form $\boldsymbol{S} = \mathbb{C}\boldsymbol{E}$ and we are in the displaced state $\boldsymbol{u}_1$ and we try to solve the increment to obtain $\boldsymbol{u}_2 = \boldsymbol{u}_1 + \Delta \boldsymbol{u}$. At the configuration 1 stresses are denoted as $\boldsymbol{S}_1$ and then

$$\boldsymbol{S}_2 = \boldsymbol{S}_1 + \Delta \boldsymbol{S} = \boldsymbol{S}_1 + \mathbb{C}\Delta \boldsymbol{E},$$

substituting it and $\delta \boldsymbol{E}$, $\Delta \boldsymbol{E}$ and $\boldsymbol{F}_2 = \boldsymbol{F}_1 + \Delta \boldsymbol{F} = \boldsymbol{F}_1 + \Delta \boldsymbol{H}$ into the VW-equation (1) gives

$$-\int_{\Omega_0} \tfrac{1}{2}[\delta \boldsymbol{H}^T(\boldsymbol{F}_1 + \Delta \boldsymbol{H}) + (\boldsymbol{F}_1^T + \Delta \boldsymbol{H}^T)\delta \boldsymbol{H}] : (\boldsymbol{S}_1 + \mathbb{C}\tfrac{1}{2}[\Delta \boldsymbol{H}^T(\boldsymbol{F}_1 + \Delta \boldsymbol{H}) + (\boldsymbol{F}_1^T + \Delta \boldsymbol{H})\Delta \boldsymbol{H}]) \, \mathrm{d}V +$$

$$+ \int_{\Omega_0} \delta \boldsymbol{u} \cdot \rho_0 \bar{\boldsymbol{b}} \, \mathrm{d}V + \int_{\partial \Omega_{t0}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A = 0$$

Linearization of virtual work - cont'd

Rearranging and neglecting all terms higher than linear in $\Delta \boldsymbol{u}$ (i.e. $\Delta \boldsymbol{H}$)-terms

$$-\int_{\Omega_0} \tfrac{1}{2}(\delta \boldsymbol{H}^T \boldsymbol{F}_1 + \boldsymbol{F}_1^T \delta \boldsymbol{H}) : \boldsymbol{S}_1 \, \mathrm{d}V + \int_{\Omega_0} \delta \boldsymbol{u} \cdot \rho_0 \bar{\boldsymbol{b}} \, \mathrm{d}V + \int_{\partial \Omega_{t0}} \delta \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A =$$

$$\int_{\Omega_0} \tfrac{1}{2}(\delta \boldsymbol{H}^T \boldsymbol{F}_1 + \boldsymbol{F}_1^T \delta \boldsymbol{H}) \mathbb{C} \tfrac{1}{2}(\Delta \boldsymbol{H}^T \boldsymbol{F}_1 + \boldsymbol{F}_1^T \Delta \boldsymbol{H}) \, \mathrm{d}V + \int_{\Omega_0} \tfrac{1}{2}(\delta \boldsymbol{H}^T \Delta \boldsymbol{H} + \Delta \boldsymbol{H}^T \delta \boldsymbol{H}) : \boldsymbol{S}_1 \, \mathrm{d}V.$$

The red part is the internal resistance force, the black is the external force and the blue gives the Jacobian matrix.

Next

Exercises on Thursday at 2 PM in class FC112.

PVW in 1-D bar example, derivation of equilibrium equations, and linearizing the virtual work equations. Using simple linear interpolation derive the FE-equations. Home assignment means to code it.