# FEM advanced course

6. exercise – Timoshenko beam element

1. Consider the cantilever beam shown in the figure below. Compute the analytical tip displacement based on Timoshenko beam model, which takes average transverse shear strains into account. The cross-section is a rectangular solid with breath b and thickness (height) t. You can use the shear correction factor $k = 5/6$, i.e. the shear rigidity is $GA_\mathrm{s} = kGA$.
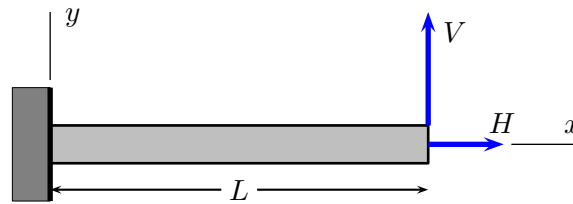
2. Compute the above cantilever beam with one simple two-node linear Timoshenko beam element with axial deformations. Set up the strain-displacement matrix $\boldsymbol{B}$ and the constitutive matrix $\boldsymbol{C}$. Use one point quadrature.

   You can use the values $L = 1$ m, $b = 20$ mm, $t = 100$ mm and 10 mm. You can use the values of aluminium with Young's modulus 70 GPa and shear modulus 26 GPa. Select the vertical load such that the vertical displacement is 2 mm and use $H = 0$.

   Calculate also the relative error in the vertical displacement?

Home assignment 6. Code the linear Timoshenko beam element. Use numerical integration and allow the choice of different quadratures and possibly on different iterpolation functions like linear and quadratic etc. Construct the code along the good practice given in Exercise 4, just repeated here, i.e. the two basic things in the FE program are the $\boldsymbol{B}$ and $\boldsymbol{C}$, which are, see slides of lecture 6

$$\boldsymbol{B} = \begin{bmatrix} N_{1,x} & 0 & 0 & N_{2,x} & 0 & 0 \\ 0 & 0 & -N_{1,x} & 0 & 0 & -N_{2,x} \\ 0 & N_{1,x} & -N_1 & 0 & N_{2,x} & -N_2 \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI & 0 \\ 0 & 0 & GA_\mathrm{s} \end{bmatrix}.$$

The degrees of freedom are arranged in the order

$$\boldsymbol{q}^{(e)} = \begin{pmatrix} u_1^{(e)} \\ v_1^{(e)} \\ \theta_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \\ \theta_2^{(e)} \end{pmatrix}.$$

Analysis case

Verify the results in the convergence plot shown in lecture 6 slide 11 with $t/L = 10^{-1}$ and 10^{-2} using 3, 10, 30, 100 elements and two point quadrature. Show in the same figure also the error for under integrated element, i.e. one point quadrature.

Write a short report on the program you have made and on the computations.

Code structure for element stiffness matrix computation

1. Set up the Gauss-points and the weights

2. Zero the element stiffness matrix and internal force vector

3. Loop over the integration points and sum up the contributions

   (a) Call the routine which gives the values of the local derivatives of the interpolation function (and possibly the interpolation functions if necessary)

   (b) Compute the Jacobian and the global derivatives of the interpolation functions

   (c) Zero the $\boldsymbol{B}$-matrix

   (d) Compute the $\boldsymbol{B}$-matrix

   (e) Set up material stiffness matrix $\boldsymbol{C}$

   (f) Multiply the values of $\boldsymbol{C}$ by the product of weight and Jacobian determinant

   (g) Add the contribution of the product $\boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B}$ to the element stiffness matrix

   (h) If the problem is nonlinear (is also valid for linear problems, the process should converge to zero residual in one iteration)

      i. Compute stresses and multiply them by the product of weight and Jacobian determinant

      ii. Compute $\boldsymbol{B}^T*$(stress resultants) and add the contribution to the element internal force vector

      iii. Compute the geometric stiffness matrix part and add the contribution to the element stiffness matrix (you need not to do this now)

The internal force vector is assembled from element contributions

$$\int_{I^{(e)}} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{\Sigma} \,\mathrm{d}x, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} N \\ M \\ Q \end{pmatrix} = \boldsymbol{C}\boldsymbol{e}, \qquad \boldsymbol{e} = \begin{pmatrix} \varepsilon \\ \kappa \\ \gamma \end{pmatrix}.$$

Solution report should be returned in Moodle before exercise 8